Intuitionistic Proofs Without Syntax

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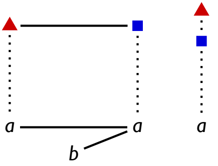
(Classical) Combinatorial Proofs



nicely coloured cograph

skew fibration

cograph

$$((a \Rightarrow b) \Rightarrow a) \Rightarrow a$$

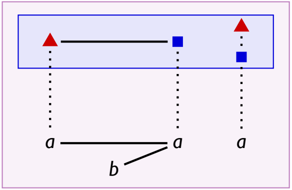
$$((\overline{a} \vee b) \wedge \overline{a}) \vee a$$

Question: What is the intuitionistic counterpart?

But first: What is a combinatorial proof?

[Hughes 2006]

… the flow of a stratified deep-inference proof



$((\bar{a} \vee b) \wedge \bar{a}) \vee a$

$\sim$

$$\dfrac{\dfrac{\top}{\bar{a} \vee a}\,a \;\wedge\; \dfrac{\top}{\bar{a} \vee a}\,a}{\dfrac{(\bar{a} \vee a) \wedge \bar{a}}{(\bar{a} \wedge \bar{a}) \vee a}\,s \;\vee\; a}\,s$$

$$\left(\dfrac{\bar{a}}{\bar{a} \vee b}\,w \;\wedge\; \bar{a}\right) \;\vee\; \dfrac{a \vee a}{a}\,c$$

nicely coloured cograph $\sim$ axiom–switch derivation

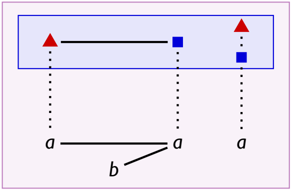
$$\dfrac{\top}{a \vee \bar{a}}\,a \qquad \dfrac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,s$$

skew fibration $\sim$ contraction–weakening derivation

$$\dfrac{A}{A \vee B}\,w \qquad \dfrac{A \vee A}{A}\,c$$

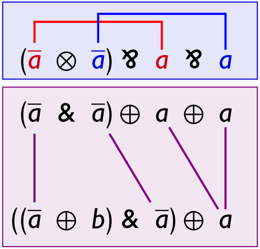
[Guglielmi et al. 1999–present]

... an MLL proof net + ALL proof net



$((\bar{a} \vee b) \wedge \bar{a}) \vee a$

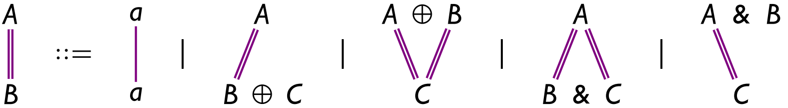
$\sim$



$((\bar{a} \vee b) \wedge \bar{a}) \vee a$

nicely coloured cograph $\sim$ MLL proof net

skew fibration $\sim$ functional ALL proof net



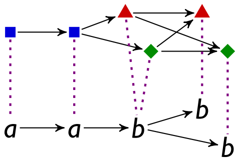
[Girard 1987, Retoré 2003, Hughes & Van Glabbeek 2005]

Classical combinatorial proofs

- Purely geometric
- Possibly canonical
- Complexity conscious (efficient (de-)sequentialization)
- Quite nice

Question: What is the intuitionistic counterpart?

Intuitionistic Combinatorial Proofs



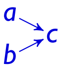
Arena net

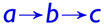
Skew fibration

Arena

$$((a \Rightarrow a) \Rightarrow b) \Rightarrow (b \wedge b)$$

Part 1: From formulas to arenas

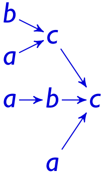




$a \Rightarrow b \Rightarrow c$

$(a \Rightarrow b) \Rightarrow c$

$(a \wedge b) \Rightarrow c$



$(a \Rightarrow b \Rightarrow c) \Rightarrow (a \Rightarrow b) \Rightarrow a \Rightarrow c$

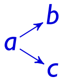
$(((a \wedge b) \Rightarrow c) \wedge (a \Rightarrow b) \wedge a) \Rightarrow c$

$a \rightarrow b$

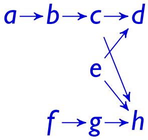
$a \rightarrow c$

$a \Rightarrow (b \wedge c)$

$(a \Rightarrow b) \wedge (a \Rightarrow c)$



$a \Rightarrow (b \wedge c)$

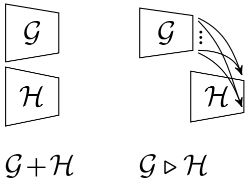


$(((a \Rightarrow b) \Rightarrow c) \wedge e) \Rightarrow (d \wedge ((f \Rightarrow g) \Rightarrow h))$

See also [McCusker 2000]

Arenas, inductively

$$
\begin{aligned}
[\![a]\!] &= \bullet^a \quad \text{(a node labelled } a\text{)} \\
[\![A \wedge B]\!] &= [\![A]\!] + [\![B]\!] \\
[\![A \Rightarrow B]\!] &= [\![A]\!] \triangleright [\![B]\!]
\end{aligned}
$$

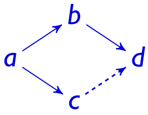


$\mathcal{G} + \mathcal{H}$: union (assuming distinct sets of vertices)

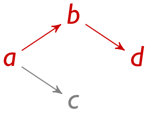
$\mathcal{G} \triangleright \mathcal{H}$: union, and connect all roots of $\mathcal{G}$ to all roots of $\mathcal{H}$

Arenas, geometrically

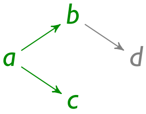
L-free: if $c \leftarrow a \rightarrow b \rightarrow d$ then $c \rightarrow d$



$(a \Rightarrow (b \wedge c)) \Rightarrow d$



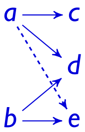
$(a \Rightarrow b) \Rightarrow d$



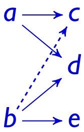
$a \Rightarrow (b \wedge c)$

Arenas, geometrically

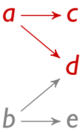
Σ-free: if $c \leftarrow a \rightarrow d \leftarrow b \rightarrow e$ then $a \rightarrow e$ or $b \rightarrow c$

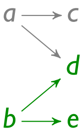


or



$a \Rightarrow (c \wedge (b \Rightarrow (d \wedge e)))$ $b \Rightarrow ((a \Rightarrow (c \wedge d)) \wedge e)$

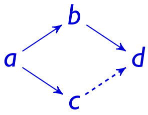




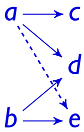
$a \Rightarrow (c \wedge d)$ $b \Rightarrow (d \wedge e)$

Arenas, geometrically

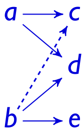
L-free:



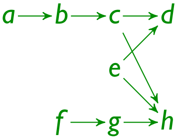
Σ-free:



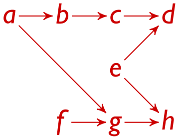
or



Example:



Non-example:



Theorem

A directed acyclic graph (DAG) represents a formula $[\![A]\!]$ if and only if it is L-free and Σ-free.

Theorem

$[\![A]\!] = [\![B]\!]$ if and only if $A \sim B$ by the isomorphisms

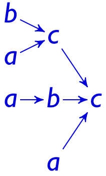
$$(A \wedge B) \Rightarrow C \;\sim\; A \Rightarrow B \Rightarrow C \qquad A \wedge B \;\sim\; B \wedge A \qquad (A \wedge B) \wedge C \;\sim\; A \wedge (B \wedge C).$$

Represent *"labelled with the same atom"* abstractly by a *partitioning*:

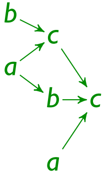
Definition

An **arena** is an L-free, Σ-free DAG with a partitioning of its vertices.

Example: S-combinator



$((a \Rightarrow b \Rightarrow c) \wedge (a \Rightarrow b)) \Rightarrow a \Rightarrow c$



$(a \Rightarrow ((b \Rightarrow c) \wedge b)) \Rightarrow a \Rightarrow c$

Part 2: From IMLL proof nets to arena nets

IMLL

Formulas

$$A ::= a \mid A \otimes B \mid A \multimap B$$

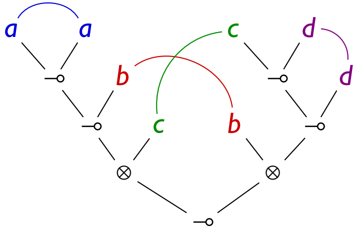
Sequent calculus:

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}$$

$$\frac{}{a \vdash a}$$

IMLL proof nets

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a\vdash a}}{\vdash a\multimap a}\qquad \dfrac{}{b\vdash b}}{(a\multimap a)\multimap b\vdash b}\qquad \dfrac{\dfrac{\dfrac{}{c\vdash c}\qquad\dfrac{}{d\vdash d}}{c\,,\ c\multimap d\vdash d}}{c\vdash (c\multimap d)\multimap d}}{(a\multimap a)\multimap b\,,\ c\vdash b\otimes((c\multimap d)\multimap d)}}{((a\multimap a)\multimap b)\otimes c\vdash b\otimes((c\multimap d)\multimap d)}}{\vdash (((a\multimap a)\multimap b)\otimes c)\multimap(b\otimes((c\multimap d)\multimap d))}$$



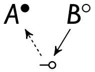
Paths & Polarity

even$^\circ$ odd$^\bullet$

In natural deduction style:

$$\dfrac{\begin{array}{c}\overline{A}^{\,x}\\ \vdots \\ B\end{array}}{A \multimap B}\multimap I,x \qquad \dfrac{A \qquad B}{A\otimes B}\otimes I \qquad \dfrac{A\multimap B \qquad A}{B}\multimap E \qquad \dfrac{A\otimes B}{A \quad B}\otimes E$$

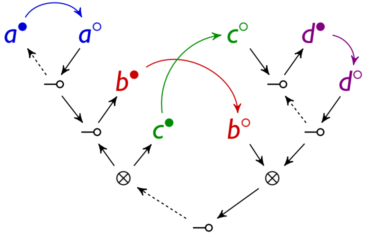
Correctness: (The *essential net* condition)

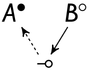
In 

every path from *A* to the root must pass *B*.

[Lamarche 2008]

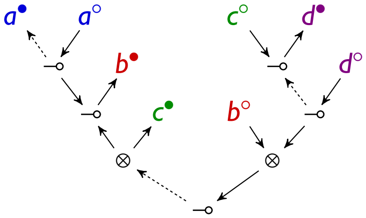
IMLL proof nets

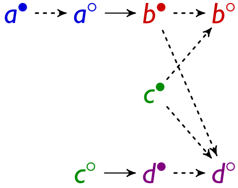


Correctness: in  every path from *A* to the root must pass *B*.

Paths in arenas

$$(((a^\bullet \multimap a^\circ) \multimap b^\bullet) \otimes c^\bullet) \multimap (b^\circ \otimes ((c^\circ \multimap d^\bullet) \multimap d^\circ))$$



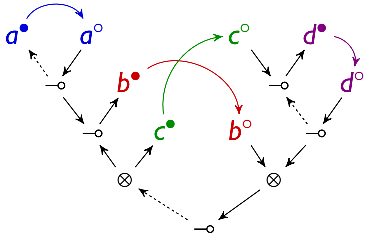


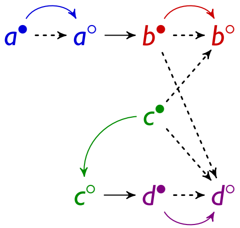
Lemma

Formula-paths $x^\circ \rightarrow^* y^\bullet$ correspond to arena-edges $x^\circ \rightarrow y^\bullet$.

Formula-paths $x^\circ \rightarrow^* \dashrightarrow \rightarrow^* y^\bullet$ correspond to arena-edges $y^\bullet \dashrightarrow x^\circ$.

An arena is *linked* if each partition is *binary* and *dual* $\{x^\bullet, x^\circ\}$ (a *link*)
The *link graph* of an arena are the *even edges* $x^\circ \rightarrow y^\bullet$ and *links* $x^\bullet \curvearrowright x^\circ$





A linked arena is *correct* if: *(Acyclicity)* the link graph is acyclic, and *(Functionality)* a rooted link path $a^\bullet \rightarrow^* r^\circ$ passes some b° with $a^\bullet \dashrightarrow b^\circ$.

Theorem

A linked arena is correct if and only if it represents an IMLL proof net.

Definition

An *arena net* is a correct linked arena.

Part 3: Skew fibrations

Contraction-weakening derivations in open deduction:

$$a \qquad \begin{array}{c}A\\ \Downarrow\\ B\end{array} \wedge \begin{array}{c}C\\ \Downarrow\\ D\end{array} \qquad \begin{array}{c}B\\ \Uparrow\\ A\end{array} \Rightarrow \begin{array}{c}C\\ \Downarrow\\ D\end{array} \qquad \begin{array}{c}A\\ \Downarrow\\ B\\ \Downarrow\\ C\end{array} \qquad \frac{A}{A \wedge A}\,c \qquad \frac{A}{1}\,w$$

But: classically contract/weaken only on disjunction — *odd* conjunction

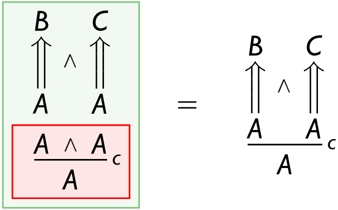
$$\begin{matrix}A\\ \Downarrow\\ B\end{matrix} \;::=\; a \;\Big|\; \begin{matrix}A\\ \Downarrow\\ B\end{matrix} \wedge \begin{matrix}C\\ \Downarrow\\ D\end{matrix} \;\Big|\; \begin{matrix}B\\ \Uparrow\\ A\end{matrix} \Rightarrow \begin{matrix}C\\ \Downarrow\\ D\end{matrix} \;\Big|\; \begin{matrix}A\\ \Downarrow\\ B\\ \Downarrow\\ C\end{matrix}$$

$$\begin{matrix}B\\ \Uparrow\\ A\end{matrix} \;::=\; a \;\Big|\; \begin{matrix}B\\ \Uparrow\\ A\end{matrix} \wedge \begin{matrix}D\\ \Uparrow\\ C\end{matrix} \;\Big|\; \begin{matrix}A\\ \Downarrow\\ B\end{matrix} \Rightarrow \begin{matrix}D\\ \Uparrow\\ C\end{matrix} \;\Big|\; \begin{matrix}C\\ \Uparrow\\ B\\ \Uparrow\\ A\end{matrix} \;\Big|\; \frac{A \wedge A}{A}\,c \;\Big|\; \frac{\mathsf{I}}{A}\,w$$

Arenas $[\![A]\!]$ give associativity, symmetry, and units for free:

$$\frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C} \qquad \frac{A \wedge B}{B \wedge A} \qquad \frac{A \wedge \mathrm{I}}{A}$$

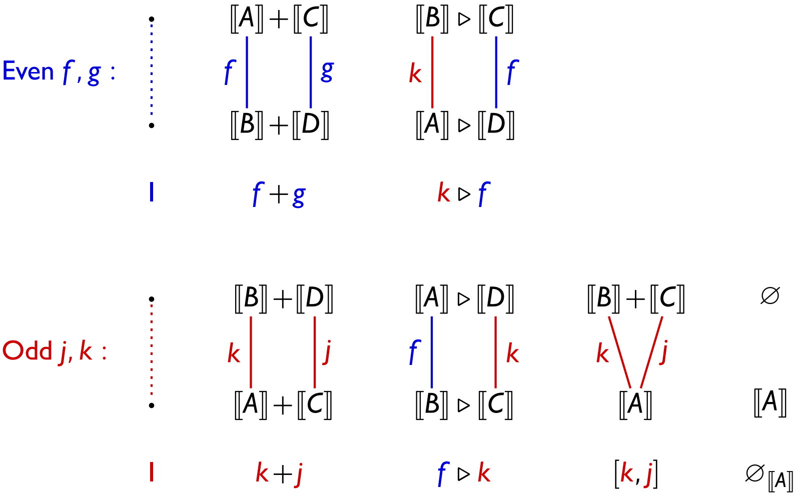
Then vertical composition is only used with contraction:



$$\begin{array}{c}A\\ \Downarrow\\ B\end{array} \; ::= \; a \;\Big|\; \begin{array}{c}A\\ \Downarrow\\ B\end{array} \wedge \begin{array}{c}C\\ \Downarrow\\ D\end{array} \;\Big|\; \begin{array}{c}B\\ \Uparrow\\ A\end{array} \Rightarrow \begin{array}{c}C\\ \Downarrow\\ D\end{array}$$

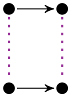
$$\begin{array}{c}B\\ \Uparrow\\ A\end{array} \; ::= \; a \;\Big|\; \begin{array}{c}B\\ \Uparrow\\ A\end{array} \wedge \begin{array}{c}D\\ \Uparrow\\ C\end{array} \;\Big|\; \begin{array}{c}A\\ \Downarrow\\ B\end{array} \Rightarrow \begin{array}{c}D\\ \Uparrow\\ C\end{array} \;\Big|\; \dfrac{\begin{array}{c}B\\ \Uparrow\\ A\end{array} \wedge \begin{array}{c}C\\ \Uparrow\\ A\end{array}}{A}\,c \;\Big|\; \dfrac{\mathbf{I}}{A}\,w$$

Skew fibrations, inductively

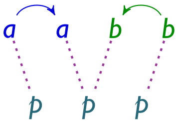


Skew fibrations, geometrically

- Preserve edges (and roots):

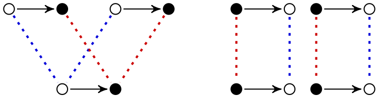


- Preserve axiom links/partitioning (but not labels!):



Skew fibrations, geometrically

Contract on odd (●) but not even (○) nodes — and their subgraphs



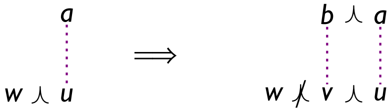
Two vertices $x \neq y$ are *conjunctively related* $x \curlywedge y$ if they *meet* at even depth (or not at all):

$$x \curlywedge y: \text{ if } \quad x \to^{n} z \,{}^{m}\!\leftarrow y \quad \text{for minimal } n, m \quad \text{then} \quad z^{\circ}$$

- Preserve conjunctive relations

Skew fibrations, geometrically

The skew lifting property:



$$\frac{1}{w} \wedge \begin{array}{c} a \\ \Downarrow \\ u \end{array} \quad \Longrightarrow \quad \left(\frac{1}{w} \Rightarrow \begin{array}{c} b \\ \Downarrow \\ v \end{array} \right) \wedge \begin{array}{c} a \\ \Downarrow \\ u \end{array}$$

Theorem

A graph homomorphism is *"(even) inductive"* if and only if it preserves edges, roots, partitioning, and conjunctive relations, and satisfies *skew lifting*.

Definition

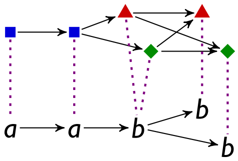
A *skew fibration* is a graph homomorphism that preserves edges, roots, pertitioning, and conjunctive relations, and satisfies *skew lifting*.

Definition

An *intuitionistic combinatorial proof* of a formula A is a skew fibration

$$f : \mathcal{A} \to [\![A]\!]$$

from an arena net $\mathcal{A}$ to the arena of A.



Arena net

Skew fibration

Arena

$$((a \Rightarrow a) \Rightarrow b) \Rightarrow (b \wedge b)$$

Intuitionistic combinatorial proofs

- Purely geometric
- Locally canonical (factor out non-duplicating permutations)
- Polynomial full completeness (efficient (de-)sequentialization)
- Quite nice