An introduction to deep inference

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Frege/Hilbert/Ackermann

Axiom schemas

$$K : A \to B \to A$$

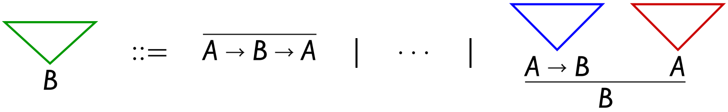
$$S : (A \to B \to C) \to (A \to B) \to A \to C$$

$$P : ((A \to B) \to A) \to A$$

plus Modus Ponens (MP): if A and $A \to B$ then B

1.	$(A \to (B \to A) \to A) \to (A \to B \to A) \to A \to A$	(S)
2.	$A \to (B \to A) \to A$	(K)
3.	$(A \to B \to A) \to A \to A$	$(MP\ 1, 2)$
4.	$A \to B \to A$	(K)
5.	$A \to A$	$(MP\ 3, 4)$

What is the structure of proofs?



- Logical consequence ($A \to B$): only at the level of formulas
- Branching (horizontal): proof-level conjunction
- Modus ponens: mixes formula-implication and proof-conjunction

(There are benefits to a simple proof structure; in particular, implementation of functional programming via *supercombinators* is simple and fast [Hughes, 1982].)

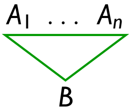
Gentzen: Natural Deduction

	introduction	elimination
conjunction	$\dfrac{A \qquad B}{A \wedge B}$	$\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$
implication	$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}$	$\dfrac{A \to B \qquad A}{B}$

Features:

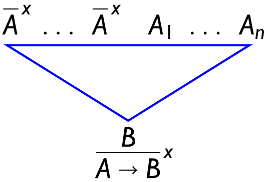
- Corresponds to natural reasoning
- Defining introduction/elimination rules for each connective
- Normalization

What is the structure of proofs?

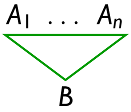


Aside: what does the implication introduction rule actually mean?

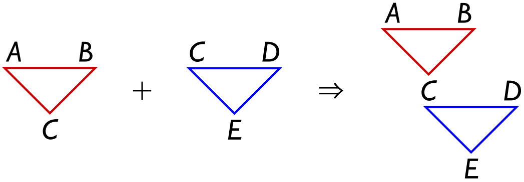
$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \to B}$$



What is the structure of proofs?



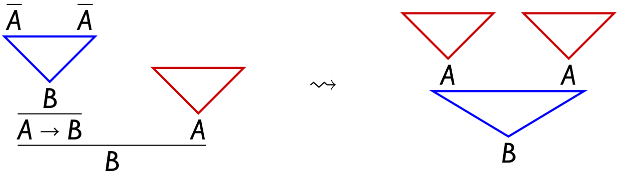
- Deriving (vertical): proof-level implication ($A \to B$)
- Branching (horizontal): proof-level conjunction ($A_1 \wedge \cdots \wedge A_n$)
- Modus ponens: mixes formula-implication and proof-conjunction
- Proof composition: implements proof-level modus ponens



No proof-level disjunction — encoded via implication and conjunction:

	introduction		elimination
disjunction	$\dfrac{A}{A \lor B}$	$\dfrac{B}{A \lor B}$	$\dfrac{A \lor B \qquad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$

Assumption closing is ad-hoc, non-trivial, and non-local



Gentzen: Sequent Calculus

Sequents: $\Gamma \vdash B$ where $\Gamma = A_1 \ \ldots \ A_n$ is a *context*

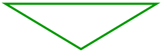
Logical rules:

	left	right
conjunction	$\dfrac{\Gamma \ A \ B \vdash C}{\Gamma \ A \wedge B \vdash C}$	$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \ \Delta \vdash A \wedge B}$
implication	$\dfrac{\Gamma \vdash A \qquad B \ \Delta \vdash C}{\Gamma \ A \rightarrow B \ \Delta \vdash C}$	$\dfrac{\Gamma \ A \vdash B}{\Gamma \vdash A \rightarrow B}$

Structural rules:

$$\frac{\Gamma \ A \ A \vdash B}{\Gamma \ A \vdash B} \qquad \frac{\Gamma \vdash B}{\Gamma \ A \vdash B} \qquad \frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \qquad A \ \Delta \vdash B}{\Gamma \ \Delta \vdash B}$$

What is the structure of proofs?



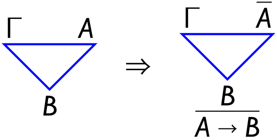
$$A_1 \cdots A_n \vdash B$$

- Deriving (vertical): proof-level implication
- Branching (horizontal): proof-level conjunction
- Sequents ($A_1 \cdots A_n \vdash B$): another proof-level implication ($A \to B$)
- Contexts ($A_1 \cdots A_n$): another proof-level conjunction ($A_1 \wedge \cdots \wedge A_n$)
- Cut-rule: mixes sequent-implication and branching-conjunction
- Implication-left: mixes formula-implication and branching-conjunction

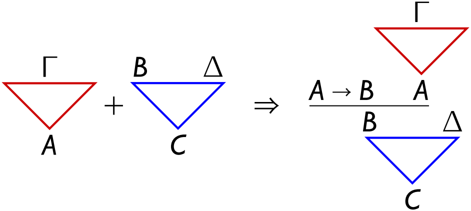
$$\frac{\Gamma \vdash A \qquad A\ \Delta \vdash B}{\Gamma\ \Delta \vdash B} \qquad\qquad \frac{\Gamma \vdash A \qquad B\ \Delta \vdash C}{\Gamma\ A \to B\ \Delta \vdash C}$$

Sequent calculus is a meta-calculus:

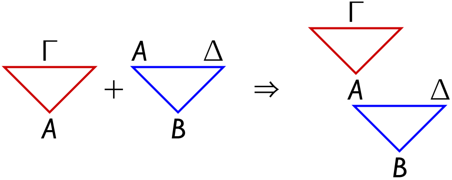
$$\frac{\Gamma\ A \vdash B}{\Gamma \vdash A \to B}$$



$$\frac{\Gamma \vdash A \qquad B\ \Delta \vdash C}{\Gamma\ A \to B\ \Delta \vdash C}$$



$$\frac{\Gamma \vdash A \qquad A\ \Delta \vdash B}{\Gamma\ \Delta \vdash B}$$



Deep Inference / Open Deduction

What is the structure of proofs?

$$\begin{array}{c} A \\ \Downarrow \\ B \end{array}$$

- Deriving (vertical): proof-level implication ($A \to B$)
- Proof composition implements proof-level modus ponens

$$\begin{array}{c} A \\ \Downarrow \\ B \end{array} \quad + \quad \begin{array}{c} B \\ \Downarrow \\ C \end{array} \quad \Rightarrow \quad \begin{array}{c} A \\ \Downarrow \\ B \\ \Downarrow \\ C \end{array}$$

A formula A is a proof with premise A and conclusion A.
Composition is associative and has formulas as unit.

Inference rules

conjunction: $\dfrac{A}{\top}$ $\dfrac{A}{A \wedge A}$ $\dfrac{A \wedge B}{B \wedge A}$ $\dfrac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C}$ $\dfrac{A \wedge \top}{A}$

implication: $\dfrac{(A \to B) \wedge A}{B}$ $\dfrac{B}{A \to (B \wedge A)}$

Horizontal composition

$$\begin{array}{c} A \\ \Downarrow \\ B \end{array} \wedge \begin{array}{c} C \\ \Downarrow \\ D \end{array} \;\Rightarrow\; \begin{array}{c} A \wedge C \\ \Downarrow \\ B \wedge D \end{array} \qquad \begin{array}{c} B \\ \Uparrow \\ A \end{array} \to \begin{array}{c} C \\ \Downarrow \\ D \end{array} \;\Rightarrow\; \begin{array}{c} B \to C \\ \Downarrow \\ A \to D \end{array}$$

Note that $A \Rightarrow B$ and $B \to C$ and $C \Rightarrow D$ gives $A \to D$

We consider any proof

$$\begin{array}{c} A \\ \Downarrow \\ B \end{array}$$

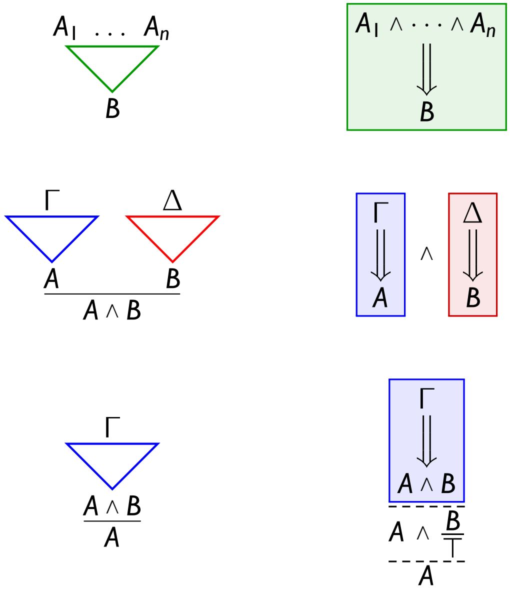
over just symmetry, associativity, unitality, and their inverses

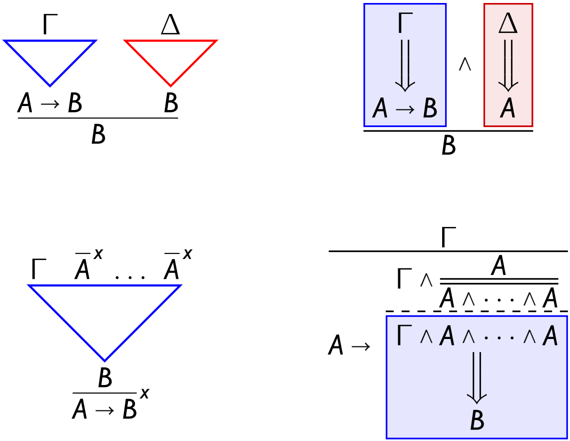
$$\frac{A \wedge B}{B \wedge A} \qquad \frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C} \qquad \frac{A \wedge \top}{A} \qquad \begin{array}{c} A \\ \Downarrow \\ B \end{array} \wedge \begin{array}{c} C \\ \Downarrow \\ D \end{array}$$

as a single monoidal coherence rule

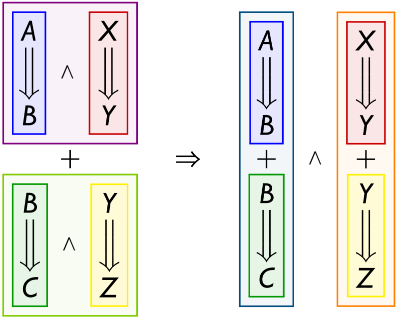
$$\frac{A}{B}$$

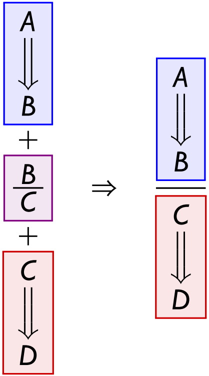
Natural Deduction | Open Deduction



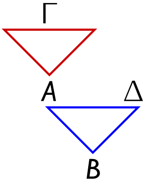

Γ
Δ
A → B
B
B
Γ
A → B
∧
Δ
A
B
Γ
A
x
. . .
A
x
B
A → B
x
Γ
Γ ∧
A
A ∧ · · · ∧ A
A →
Γ ∧ A ∧ · · · ∧ A
B

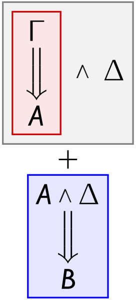
Composition in detail



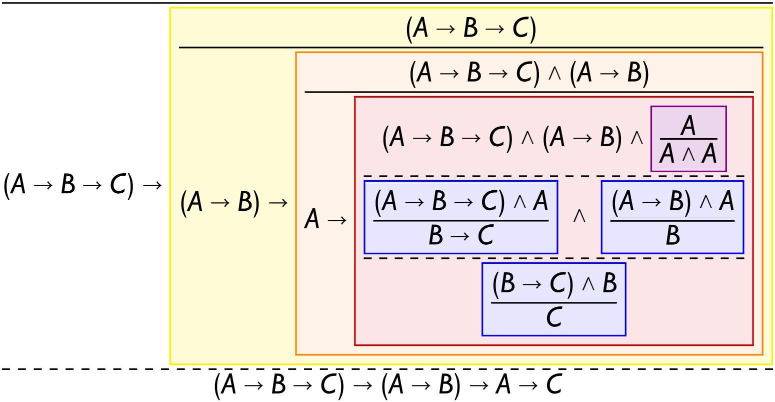


[Guglielmi, Gundersen, Parigot, 2010]


Γ
A
Δ
B


Γ
A
∧ Δ
+
A ∧ Δ
B

$$\dfrac{\top}{A \to \dfrac{\top \wedge A}{A}} \qquad \Rightarrow \qquad \dfrac{}{A \to A}$$


(A → B → C)
(A → B → C) ∧ (A → B)
(A → B → C) ∧ (A → B) ∧ A / A ∧ A
(A → B → C) → (A → B) → A → (A → B → C) ∧ A / B → C ∧ (A → B) ∧ A / B
(B → C) ∧ B / C
(A → B → C) → (A → B) → A → C

Disjunction

Inference rules

disjunction:

$$\frac{\bot}{A} \qquad \frac{A \vee A}{A} \qquad \frac{A \vee B}{B \vee A} \qquad \frac{A \vee (B \vee C)}{(A \vee B) \vee C} \qquad \frac{A \vee \bot}{A}$$

Horizontal composition

$$\begin{array}{c} A \\ \Downarrow \\ B \end{array} \vee \begin{array}{c} C \\ \Downarrow \\ D \end{array} \quad \Rightarrow \quad \begin{array}{c} A \vee C \\ \Downarrow \\ B \vee D \end{array}$$

Question Isn't this just categorical logic?

Answer Yes, it is — if you look only at what syntax is used,
but not the motivations for doing so.

~~Categorical logic does not consider computation~~

Categorical logic considers the result of computation, but not the process. This is because it considers all proof/term manipulations as equalities. Attempts to relax this via higher or enriched categories are generally not convincing.

Deep inference investigates the process and complexities of normalization. It works extremely well for classical logic (where the semantics collapses).

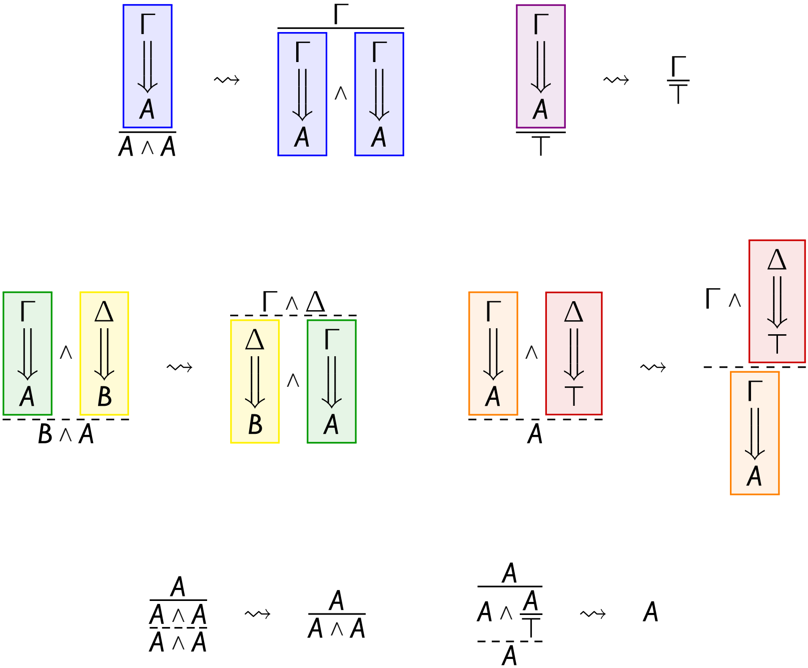
What is the structure of proofs?

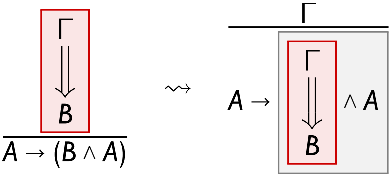
$$\begin{array}{c} A \\ \Downarrow \\ B \end{array}$$

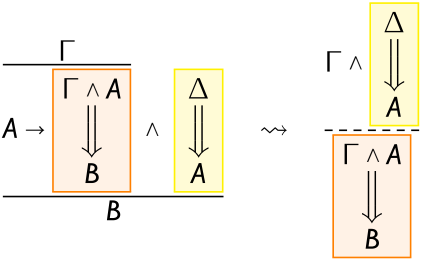
- Deriving (vertical): proof-level implication ($A \to B$)
- Proof composition implements proof-level modus ponens

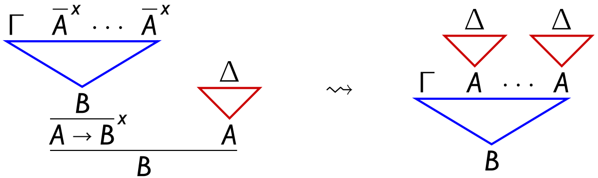
- Proof-level conjunction, implication, disjunction the same as formula-level
- Defining rules for the logical operations that are not ad-hoc
- Formula-level modus ponens for formula-implication and formula-conjunction
- Not a meta-calculus

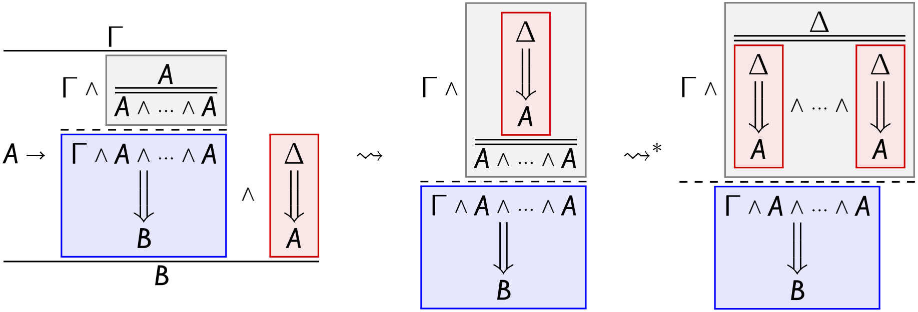
Normalization

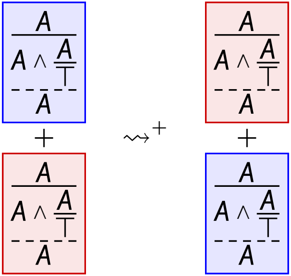


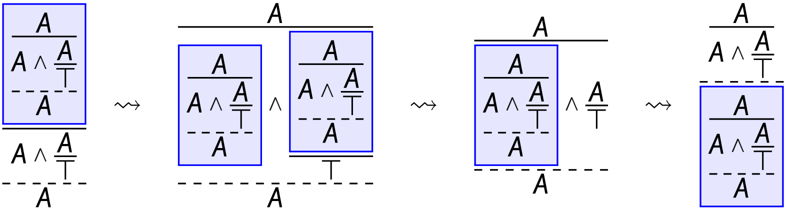

Γ
B
A → (B ∧ A)
⇝
Γ
A →
Γ
B
∧ A


Γ
A →
Γ ∧ A
B
∧
Δ
A
B
⇝
Γ ∧
Δ
A
Γ ∧ A
B


Γ
A x … A x
B
A → B x
Δ
A
B
⇝
Δ Δ
Γ A … A
B

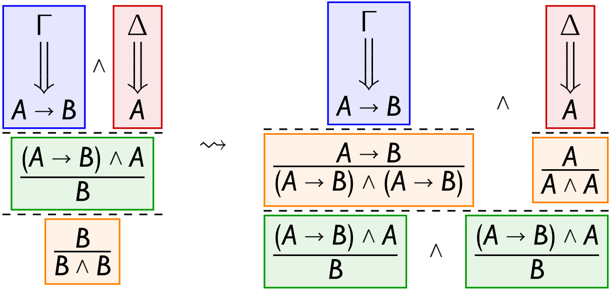

Γ
Γ ∧
A
A ∧ … ∧ A
A →
Γ ∧ A ∧ … ∧ A
B
∧
Δ
A
B
⇝
Γ ∧
Δ
A
A ∧ … ∧ A
Γ ∧ A ∧ … ∧ A
B
⇝*
Γ ∧
Δ
Δ
A
∧ … ∧
Δ
A
Γ ∧ A ∧ … ∧ A
B

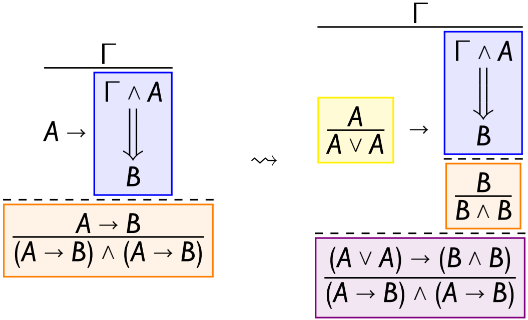




[Brünnler, McKinley, 2008]

Atomic duplication


Γ
A → B
∧
Δ
A
(A → B) ∧ A
B
B
B ∧ B
⇝
Γ
A → B
∧
Δ
A
A → B
(A → B) ∧ (A → B)
A
A ∧ A
(A → B) ∧ A
B
∧
(A → B) ∧ A
B



Γ
Γ ∧ A
A →
B
A → B
(A → B) ∧ (A → B)
Γ
Γ ∧ A
A
A ∨ A
→
B
B
B ∧ B
(A ∨ A) → (B ∧ B)
(A → B) ∧ (A → B)

Medial

$$\boxed{\frac{(A \vee B) \to (C \wedge D)}{(A \to C) \wedge (B \to D)}}$$

$$(A \vee B) \to (C \wedge D) \quad \cong \quad \begin{array}{l} (A \to C) \quad \wedge \quad (A \to D) \quad \wedge \\ (B \to C) \quad \wedge \quad (B \to D) \end{array}$$

$$\boxed{\frac{A \to B}{(A \to B) \wedge (A \to B)}} \quad \rightsquigarrow \quad \begin{array}{c} \boxed{\frac{A}{A \vee A}} \to \boxed{\frac{B}{B \wedge B}} \\ \hdashline \boxed{\frac{(A \vee A) \to (B \wedge B)}{(A \to B) \wedge (A \to B)}} \end{array}$$

$$\boxed{\frac{A \vee A}{A}} \qquad \begin{array}{ccc} B & & C \\ \Uparrow & \to & \Downarrow \\ A & & D \end{array}$$

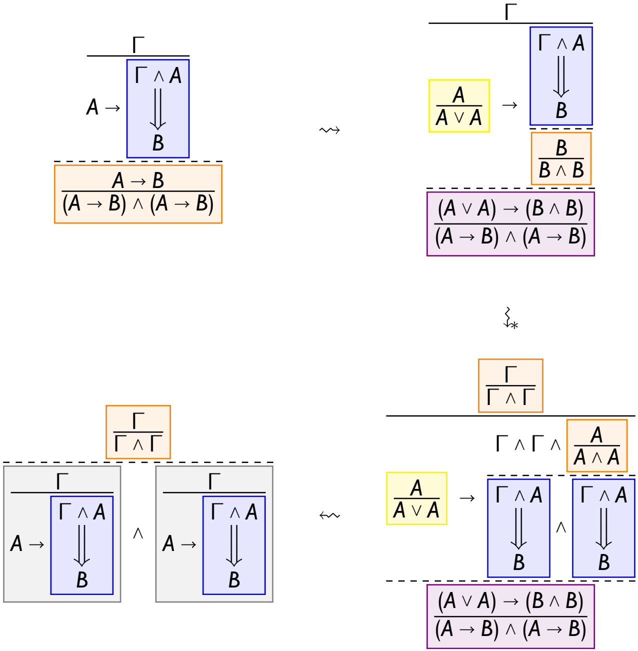
$$\frac{(A \lor B) \to (C \land D)}{(A \to C) \land (B \to D)} \qquad \frac{A \to B}{(A \to B) \land (A \to B)} \rightsquigarrow \begin{array}{c} \dfrac{A}{A \lor A} \to \dfrac{B}{B \land B} \\ \cdots\cdots\cdots \\ \dfrac{(A \lor A) \to (B \land B)}{(A \to B) \land (A \to B)} \end{array}$$

$$\frac{(A \land B) \lor (C \land D)}{(A \lor C) \land (B \lor D)} \qquad \begin{array}{c} \dfrac{A \lor A}{A} \\ \cdots\cdots \\ \dfrac{A}{A \land A} \end{array} \rightsquigarrow \begin{array}{c} \dfrac{A}{A \land A} \lor \dfrac{A}{A \land A} \\ \cdots\cdots\cdots \\ \dfrac{(A \land A) \lor (A \land A)}{(A \lor A) \land (A \lor A)} \\ \cdots\cdots\cdots \\ \dfrac{A \lor A}{A} \land \dfrac{A \lor A}{A} \end{array}$$

$$\frac{(A \to B) \lor (C \to D)}{(A \land C) \to (B \lor D)} \qquad \frac{(A \to B) \lor (A \to B)}{A \to B} \rightsquigarrow \begin{array}{c} \dfrac{(A \to B) \lor (A \to B)}{(A \land A) \to (B \lor B)} \\ \cdots\cdots\cdots \\ \dfrac{A \land A}{A} \to \dfrac{B \lor B}{B} \end{array}$$

[Tiu, 2006]

$$\dfrac{\dfrac{\dfrac{A}{A \vee A} \quad \rightarrow \quad \dfrac{A}{A \wedge A}}{}}{\dfrac{(A \vee A) \rightarrow (A \wedge A)}{(A \rightarrow A) \wedge (A \rightarrow A)}} \quad \rightsquigarrow \quad \dfrac{}{A \rightarrow A} \;\wedge\; \dfrac{}{A \rightarrow A}$$



[Gundersen, Heijltjes, Parigot, 2013]

Thank you!

For everything deep-inference go to http://alessio.guglielmi.name/res/cos