Complexity bounds for sum–product logic via additive proof nets and Petri nets

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and
Dominic Hughes**

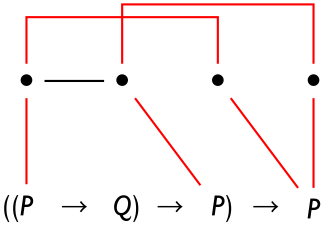
LICS, Kyoto, 6 July 2015

* University of Bath

** Stanford University

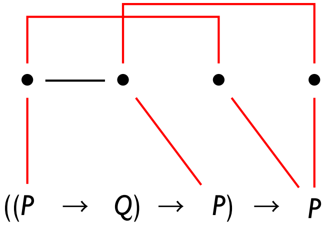
Combinatorial proofs

Purely geometric proofs for propositional classical logic



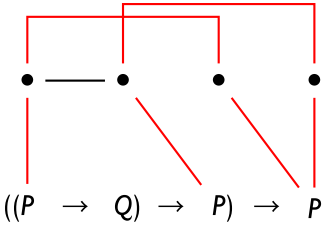
[DH, Proofs without syntax, 2005]

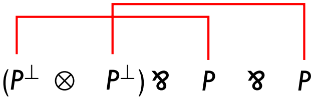
Combinatorial proofs



Combinatorial proofs

a multiplicative proof net





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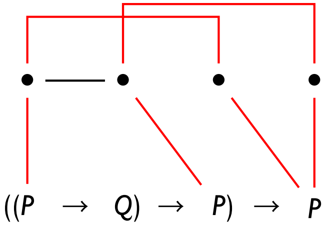
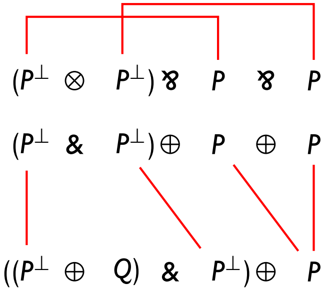
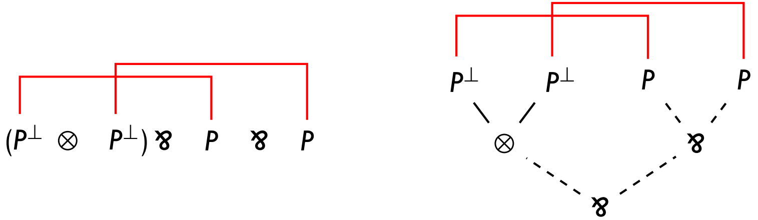




an additive proof net (that must be *functional*)

Multiplicative proof nets



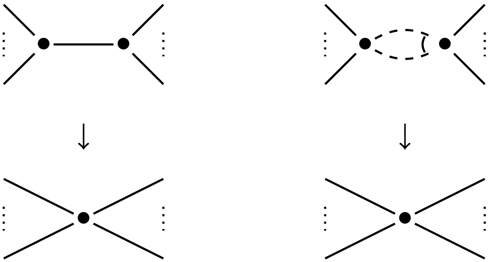
Correctness

- A switching chooses one edge of each (⅋)
- Every switching must produce a tree (connected acyclic graph)

[Girard 1987, Danos & Regnier 1989]

Contractibility

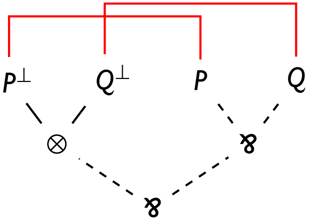
1. Start from an unlabelled graph with paired ⅋-edges
2. Contract by:

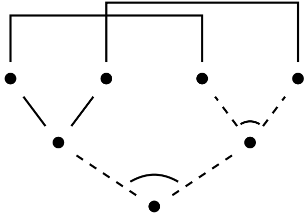


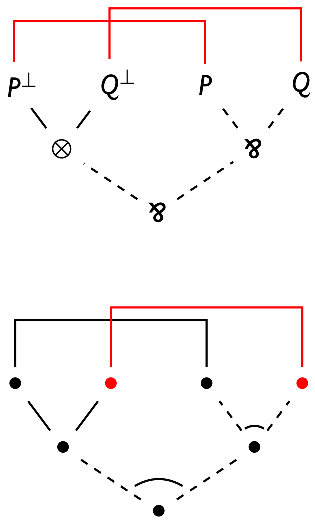
3. Correct $\Leftrightarrow$ contracts to a single point

Implemented in linear time via union–find

[Danos 1990, Guerrini 1999]


$P^{\perp}$
$Q^{\perp}$
P
Q
⊗
⅋
⅋

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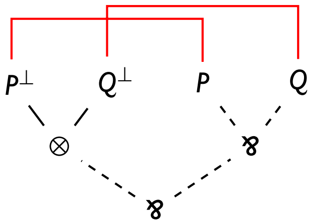
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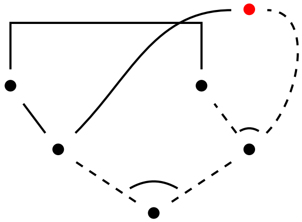

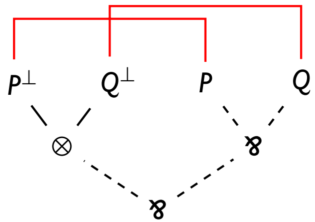
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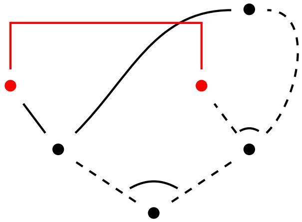

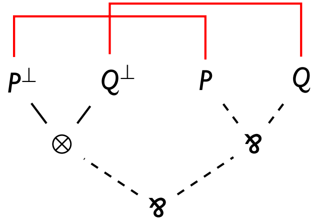
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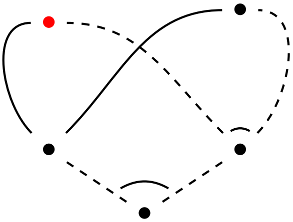

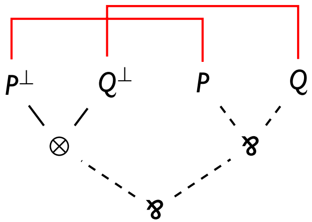

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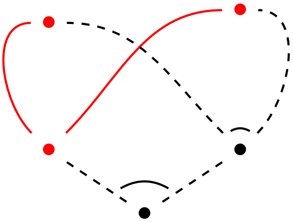

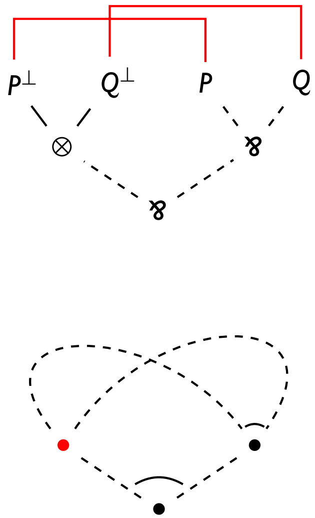

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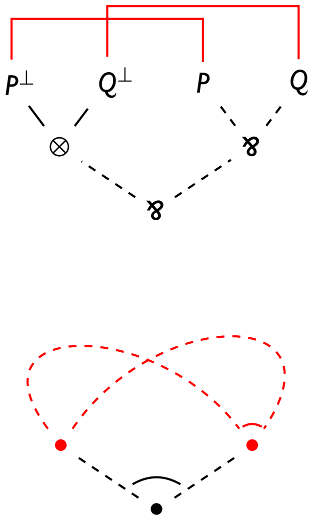
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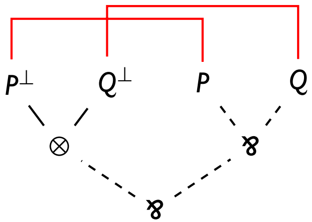
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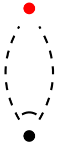

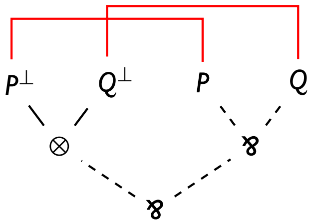
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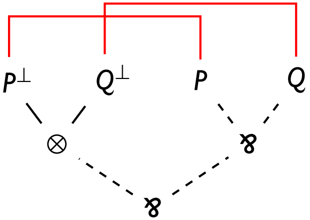

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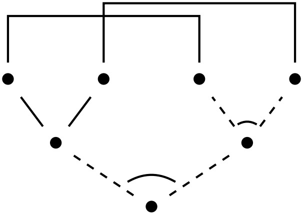
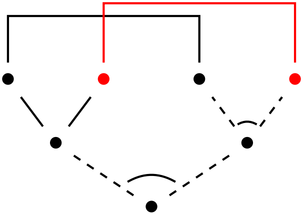


$\overline{Q^{\perp}, Q}$



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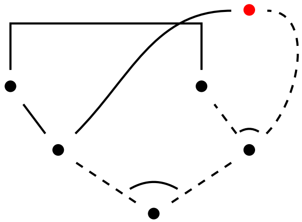
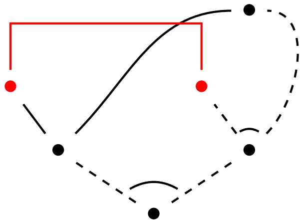
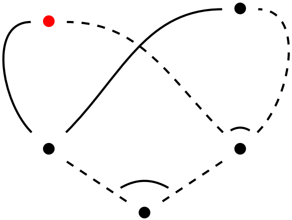


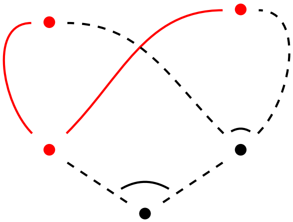
$$\overline{P^{\perp}, P} \qquad \overline{Q^{\perp}, Q}$$



$\overline{P^{\perp}, P}$ $\overline{Q^{\perp}, Q}$



$$\dfrac{\dfrac{}{P^{\perp}, P} \qquad \dfrac{}{Q^{\perp}, Q}}{(P^{\perp} \otimes Q^{\perp}), P, Q}$$



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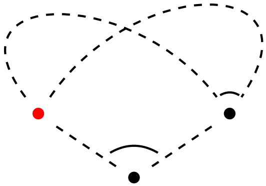

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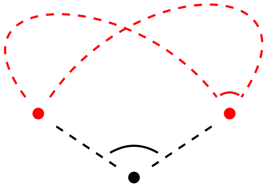
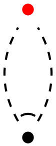
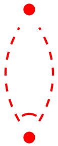


$$\dfrac{\dfrac{\dfrac{}{P^{\perp}, P} \qquad \dfrac{}{Q^{\perp}, Q}}{(P^{\perp} \otimes Q^{\perp}), P, Q}}{(P^{\perp} \otimes Q^{\perp}), P \parr Q}$$



$$\dfrac{\dfrac{\dfrac{\dfrac{}{P^{\perp}, P} \qquad \dfrac{}{Q^{\perp}, Q}}{(P^{\perp} \otimes Q^{\perp}), P, Q}}{(P^{\perp} \otimes Q^{\perp}), P \parr Q}}{(P^{\perp} \otimes Q^{\perp}) \parr P \parr Q}$$



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For multiplicative linear logic:

- We have contractibility as abstract sequentialization
- Proof net correctness is linear-time decidable

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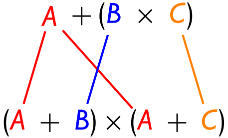
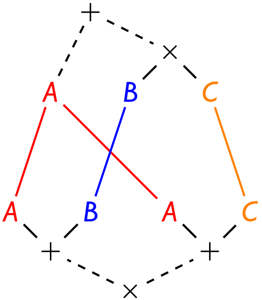
- Proof net correctness is linear-time decidable

What about additive linear logic?

Additive proof nets

(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)





Correctness

- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]

Additive proof nets

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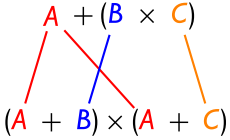

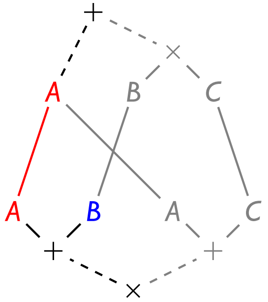

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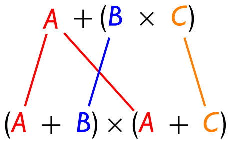

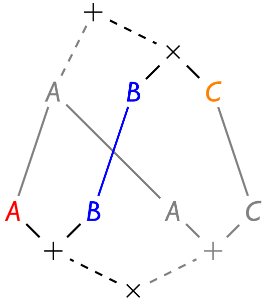

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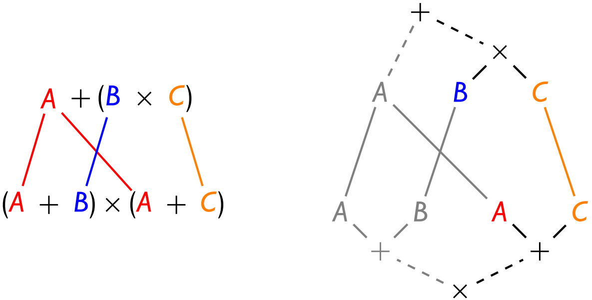

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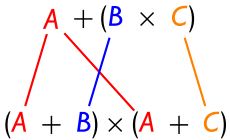

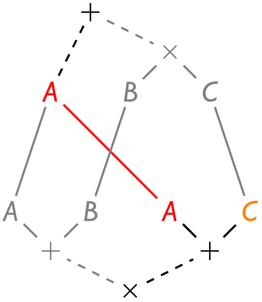

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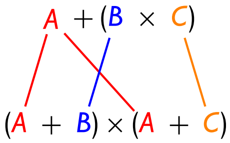

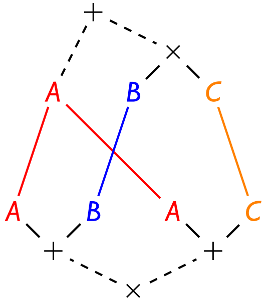

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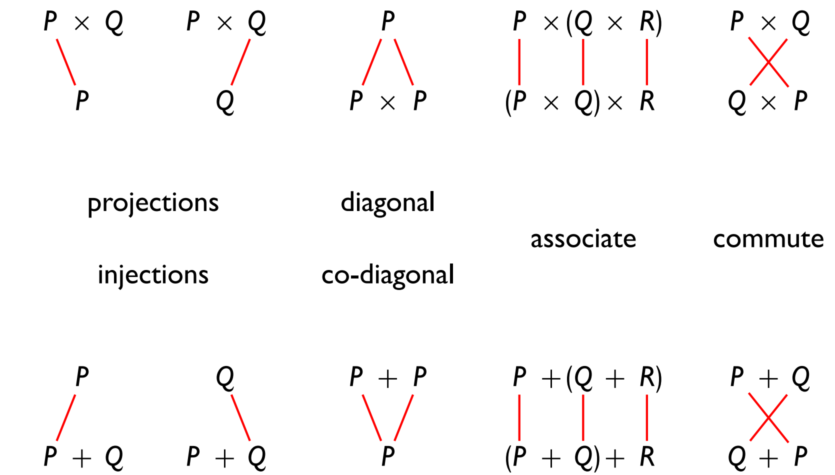
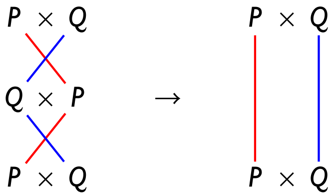


P × Q
P
P × Q
Q
projections
P
P × P
diagonal
P ×(Q × R)
(P × Q)× R
associate
P × Q
Q × P
commute
injections
P
P + Q
Q
P + Q
co-diagonal
P + P
P
P +(Q + R)
(P + Q)+ R
P + Q
Q + P

Composition is path composition



Additive linear logic / sum–product logic

Formulae:

$$A, B, C \quad ::= \quad P \quad | \quad A + B \quad | \quad A \times B$$

Sequents: $A \vdash B$ (or $\vdash A^{\perp}, B$ where $(\cdot)^{\perp}$ is DeMorgan dualization)

Sequent calculus:

$$\frac{}{P \vdash P} \qquad \frac{A \vdash B_i}{A \vdash B_0 + B_1} \qquad \frac{A \vdash B \qquad A \vdash C}{A \vdash B \times C}$$

$$\frac{A \vdash C \qquad B \vdash C}{A + B \vdash C} \qquad \frac{A_i \vdash B}{A_0 \times A_1 \vdash B}$$

Additive proof nets

A linking for a sequent $A \vdash B$ is a set of links C — D:

- C is a subformula of A
- D is a subformula of B
- C — D is an axiom link if C and D are the same atom P

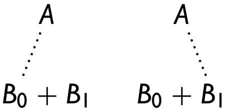
Correctness

- A resolution deletes one child of each $+$ in A and $\times$ in B
- A linking is discrete if every resolution contains one link

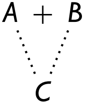
A proof net is a discrete axiom linking

$$\frac{}{P \vdash P}$$

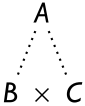
$$\frac{A \vdash B_i}{A \vdash B_0 + B_1}$$



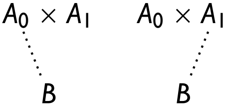
$$\frac{A \vdash C \qquad B \vdash C}{A + B \vdash C}$$



$$\frac{A \vdash B \qquad A \vdash C}{A \vdash B \times C}$$



$$\frac{A_i \vdash B}{A_0 \times A_1 \vdash B}$$

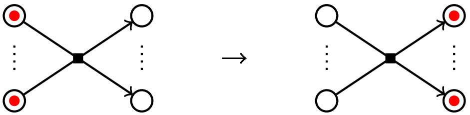


Petri nets

A model of concurrent computation

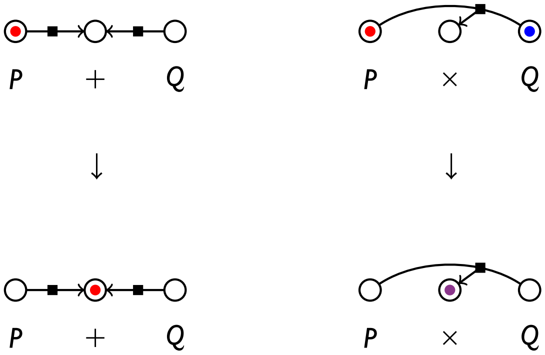
- A set $\mathcal{P}$ of places ○
- A set $\twoheadrightarrow$ of transitions $S \twoheadrightarrow T$ where $S, T \subseteq \mathcal{P}$
- A configuration, a set $M \subseteq \mathcal{P}$ of places that store a token •

Firing: if $S \twoheadrightarrow T$ and $S \subseteq M$ then $M \to (M \setminus S) \cup T$



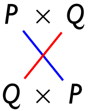
(Models where places store multiple tokens have multisets S, T, and M)

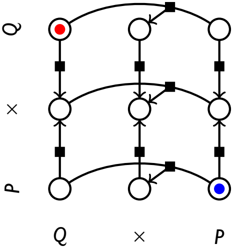
Encoding $+$ and $\times$



The nets $\mathcal{N}(P+Q)$ and $\mathcal{N}(P \times Q)$

Encoding a proof net





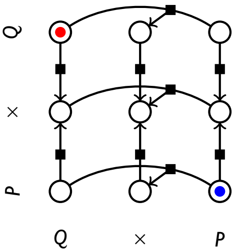
The net $\mathcal{N}(A \vdash B)$ is the cartesian product $\mathcal{N}(A^{\perp}) * \mathcal{N}(B)$
Places: $\mathcal{P} = \mathcal{P}_A \times \mathcal{P}_B$
Transitions:

$$\{p\} \times S_B \quad \twoheadrightarrow \quad \{p\} \times T_B \qquad (p \in \mathcal{P}_A,\ S_B \twoheadrightarrow T_B)$$

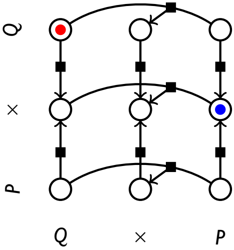
$$S_A \times \{q\} \quad \twoheadrightarrow \quad T_A \times \{q\} \qquad (S_A \twoheadrightarrow T_A,\ q \in \mathcal{P}_B)$$

Configuration: a token in (p, q) for each axiom link $P - Q$

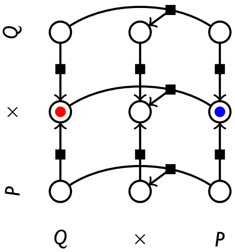
Firing a proof net



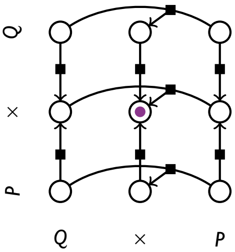
Firing a proof net



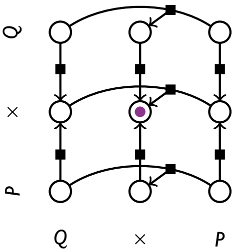
Firing a proof net



Firing a proof net

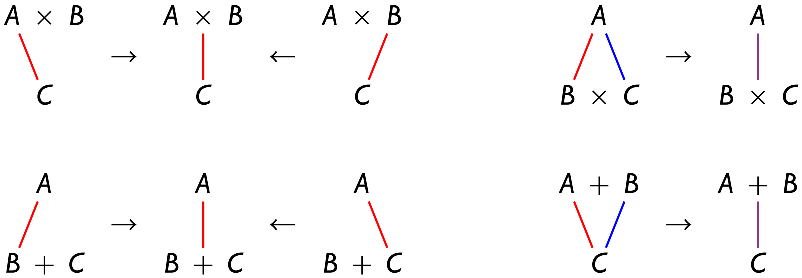


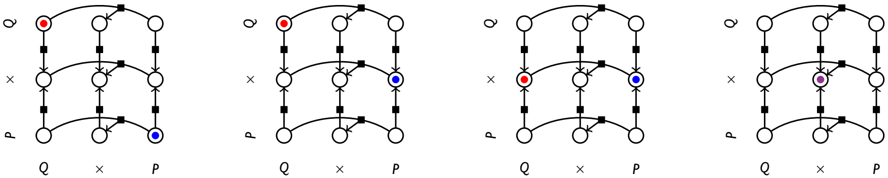
Firing a proof net

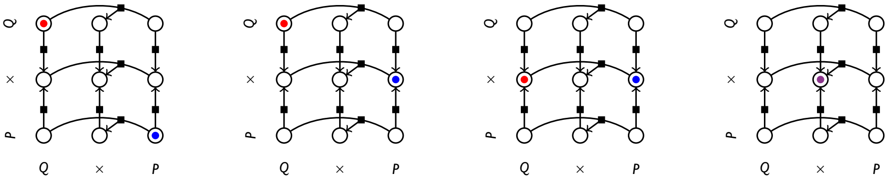


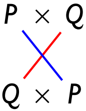
The net is correct if a single token at the root remains

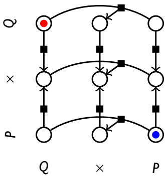
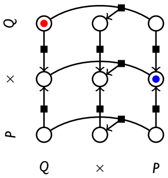
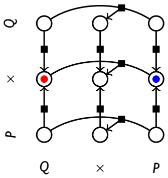
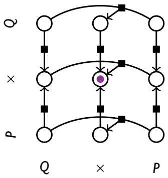
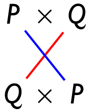
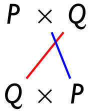
Coalescence

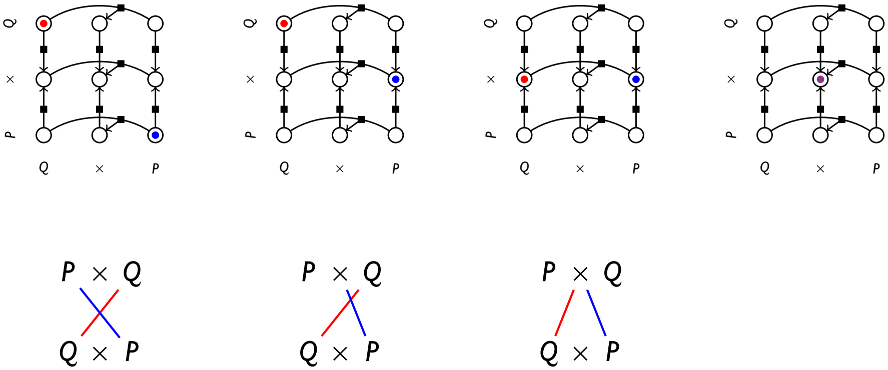


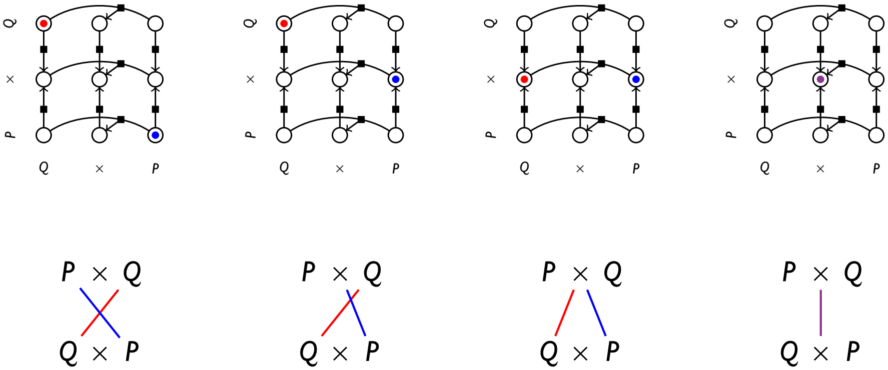

Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P


Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P
Q
×
P

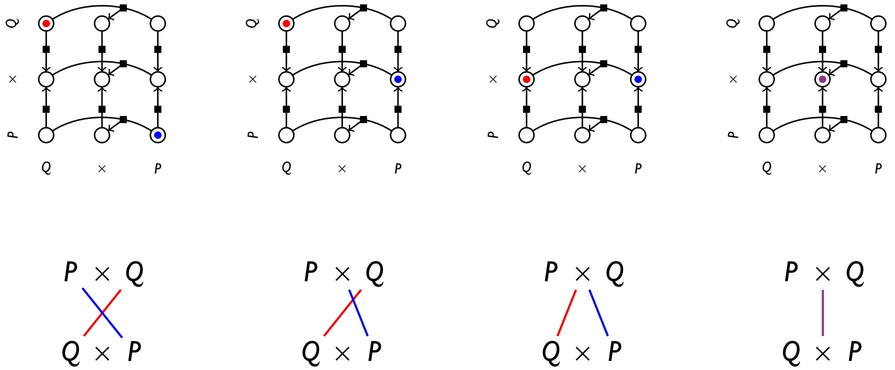

P × Q
Q × P


Q
×
P
Q × P

Q
×
P
Q × P

Q
×
P
Q × P

Q
×
P
Q × P

P × Q
Q × P

P × Q
Q × P

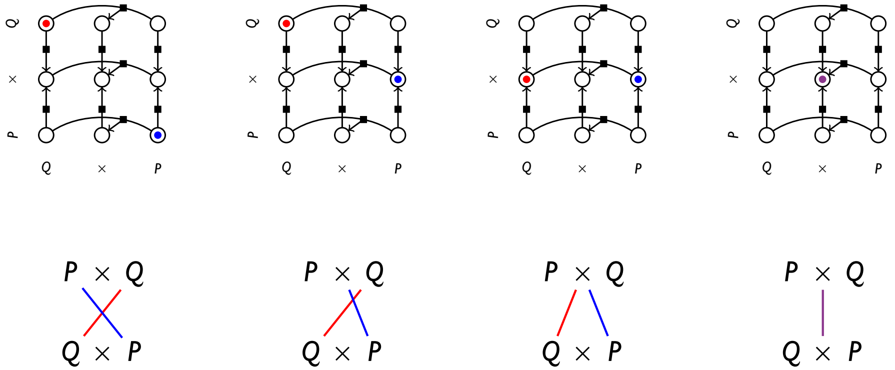

Q
×
P
Q × P
P × Q
Q × P



Q × P
Q × P
P × Q
Q × P
Q × P
Q × P
P × Q
Q × P
Q × P
Q × P
P × Q
Q × P
Q × P
Q × P
P × Q
Q × P

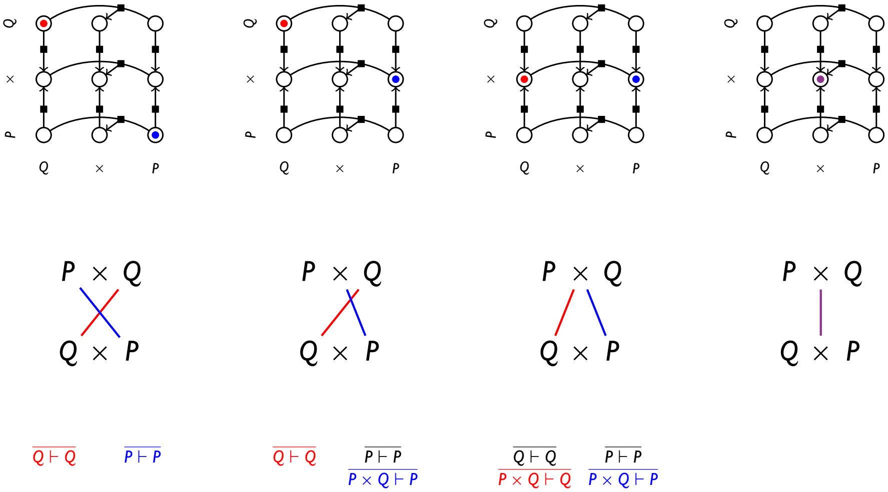


$\overline{Q \vdash Q}$ $\overline{P \vdash P}$

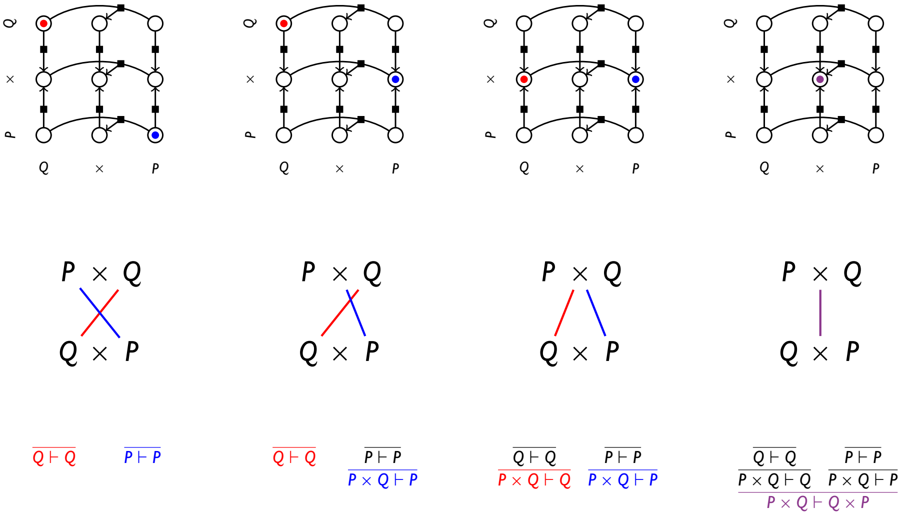


$\overline{Q \vdash Q}$ $\quad\quad$ $\overline{P \vdash P}$

$\overline{Q \vdash Q}$ $\quad\quad$ $\dfrac{\overline{P \vdash P}}{P \times Q \vdash P}$



Q × P
P × Q
Q × P
Q ⊢ Q
P ⊢ P
P × Q ⊢ P
P × Q ⊢ Q



Q × P
P × Q
Q × P
Q ⊢ Q
P ⊢ P
P × Q ⊢ P
P × Q ⊢ Q
P × Q ⊢ Q × P

Results

Theorem

Coalescence for $L : A \vdash B$ gives $\{A — B\}$ if and only if L is discrete

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Theorem

Coalescence for $L : A \vdash B$ gives $\{A — B\}$ if and only if L is discrete

Theorem

Correctness of an additive proof net $L : A \vdash B$ is decidable in time

$$\mathcal{O}(|A| \times |B|)$$

Results

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Coalescence for $L : A \vdash B$ gives $\{A — B\}$ if and only if L is discrete

Theorem
Correctness of an additive proof net $L : A \vdash B$ is decidable in time

$$\mathcal{O}(|A| \times |B|)$$

Theorem
Correctness of an additive proof net $L : A \vdash B$ is decidable in time

$$\mathcal{O}\big(|L| \times (dA + dB) \times max(log|A|,\ log|B|)\big)$$

For additive linear logic:

- We have coalescence as abstract sequentialization
- Proof net correctness is (almost) linear-time decidable

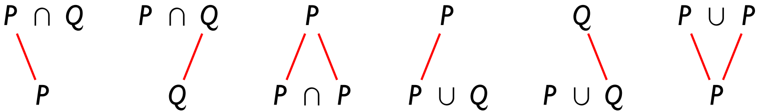
Remark

The set of subsets of a set X ordered by inclusion ($\subseteq$)

- Is a free distributive lattice: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Models ALL:

$$\begin{aligned} A \vdash B \quad &\Rightarrow \quad A \subseteq B \\ A \times B \quad &\Rightarrow \quad A \cap B \\ A + B \quad &\Rightarrow \quad A \cup B \end{aligned}$$

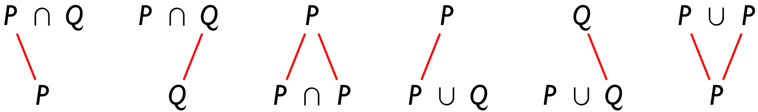
- Correctness: every resolution contains at least one link



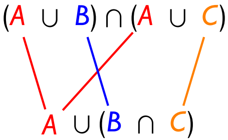
Remark

The set of subsets of a set X ordered by inclusion ($\subseteq$)

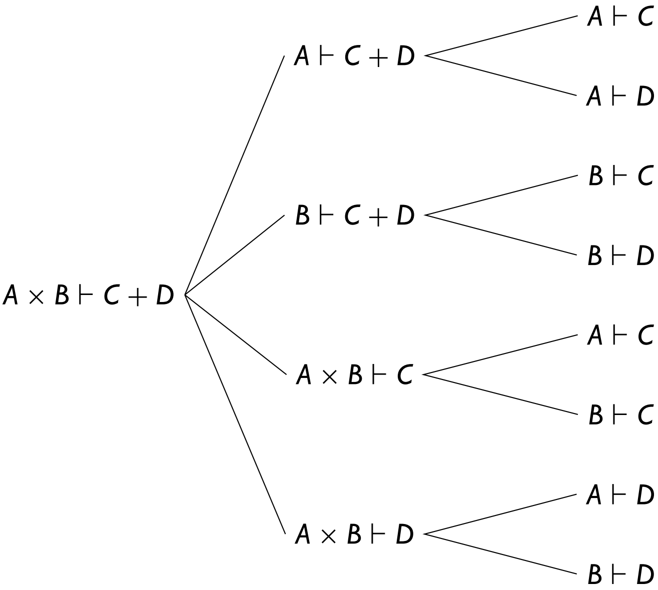
- Is a free distributive lattice: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Models ALL:
$$\begin{array}{lcl} A \vdash B & \Rightarrow & A \subseteq B \\ A \times B & \Rightarrow & A \cap B \\ A + B & \Rightarrow & A \cup B \end{array}$$
- Correctness: every resolution contains at least one link



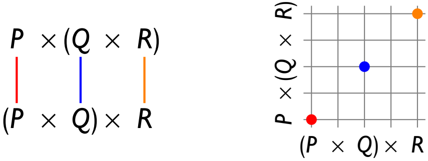
But distributivity destroys coalescence:

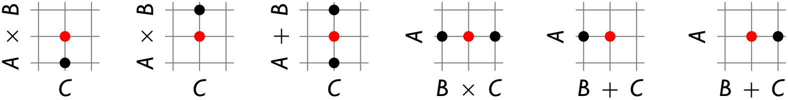


Proof search

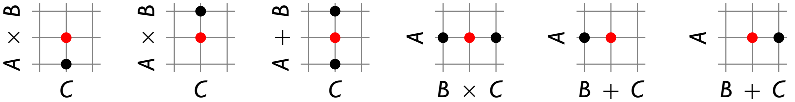


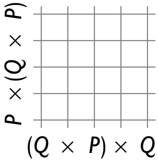
$A \times B \vdash C + D$
$A \vdash C + D$
$B \vdash C + D$
$A \times B \vdash C$
$A \times B \vdash D$
$A \vdash C$
$A \vdash D$
$B \vdash C$
$B \vdash D$
$A \vdash C$
$B \vdash C$
$A \vdash D$
$B \vdash D$

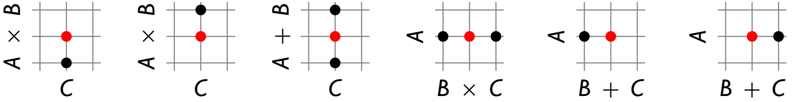

P × (Q × R)
(P × Q) × R
P × (Q × R)
(P × Q) × R

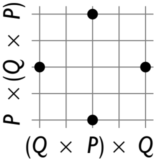


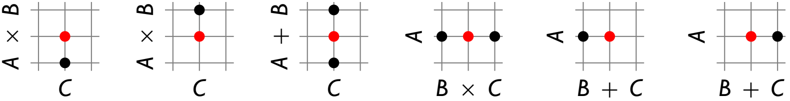
A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

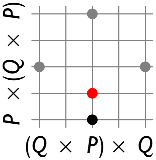

A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

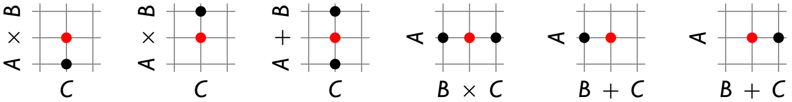

P × (Q × P)
(Q × P) × Q


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

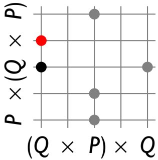

P × (Q × P)
(Q × P) × Q


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

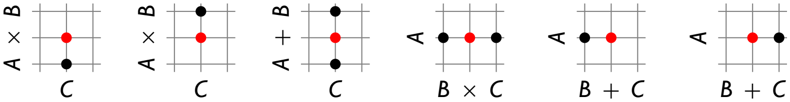

P × (Q × P)
(Q × P) × Q

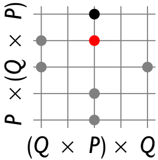


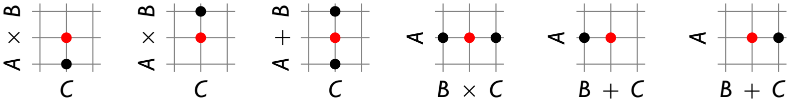
A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

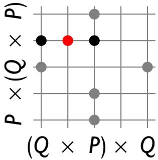


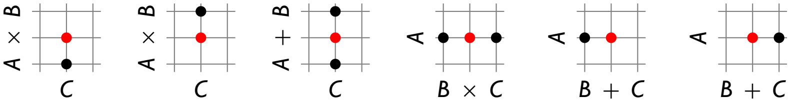
P × (Q × P)
(Q × P) × Q


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C


P × (Q × P)
(Q × P) × Q

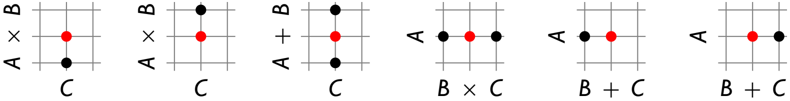

A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

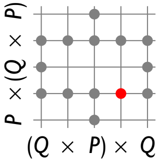

P × (Q × P)
(Q × P) × Q

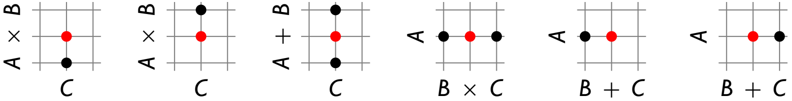


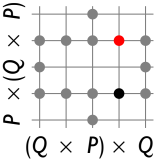
A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

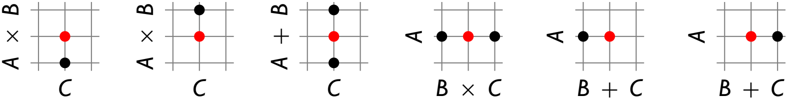
. . .

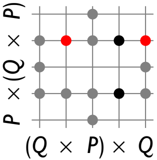

A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

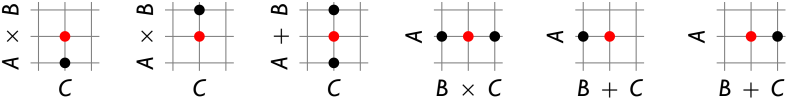

P × (Q × P)
(Q × P) × Q

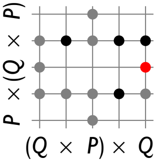

A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

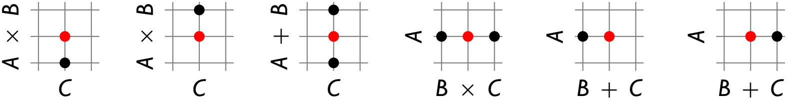

P × (Q × P)
(Q × P) × Q

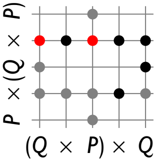

A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

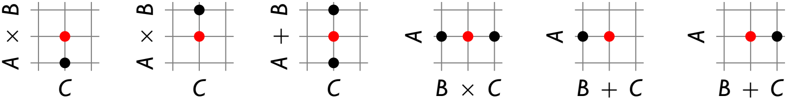

P × (Q × P)
(Q × P) × Q

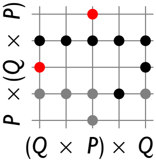

A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

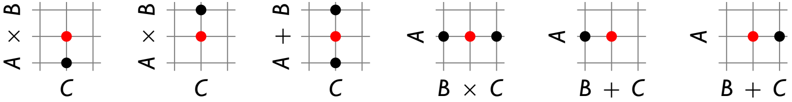

P × (Q × P)
(Q × P) × Q


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

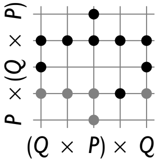

P × (Q × P)
(Q × P) × Q


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C

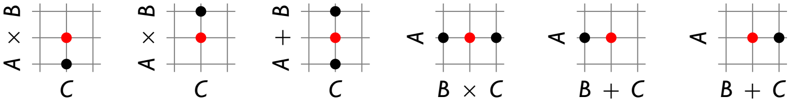

P × (Q × P)
(Q × P) × Q

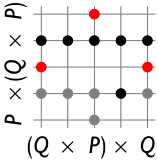


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C



P × (Q × P)
(Q × P) × Q


A × B
C
A × B
C
A + B
C
A
B × C
A
B + C
A
B + C


P × (Q × P)
(Q × P) × Q

More results

- Proof search on $A \vdash B$ is $\mathcal{O}(|A| \times |B|)$

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For additive linear logic with units:

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- Correctness of $L : A \vdash B$ is $\mathcal{O}(|A| \times |B|)$

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For first-order additive linear logic:

- Proof search is NP-complete

Further work

- First-order proofs without syntax
- Coalescence for MALL-nets
- Proof search in classical logic