Complexity bounds for sum–product logic via additive proof nets and Petri nets

Willem Heijltjes*
and
Dominic Hughes**

LICS, Kyoto, 6 July 2015

* University of Bath
** Stanford University
Combinatorial proofs

Purely geometric proofs for propositional classical logic

\[(P \rightarrow Q) \rightarrow (P) \rightarrow P\]

[DH, Proofs without syntax, 2005]
Combinatorial proofs

\[(P \rightarrow Q) \rightarrow (P \rightarrow P) \rightarrow P\]
Combinatorial proofs

a multiplicative proof net

\[ ((P \rightarrow Q) \rightarrow P) \rightarrow P \]

\[ (P \perp \otimes P \perp) \otimes P \otimes P \]

\[ P \]
Combinatorial proofs

a multiplicative proof net

\[(((P \rightarrow Q) \rightarrow P) \rightarrow P) \rightarrow P\]

an additive proof net (that must be functional)
Multiplicative proof nets

\[(P\perp \otimes P\perp) \& P \& P\]

Correctness

- A **switching** chooses one edge of each (\&)
- Every switching must produce a **tree** (connected acyclic graph)

[Girard 1987, Danos & Regnier 1989]
Contractibility

1. Start from an unlabelled graph with paired 8-edges

2. Contract by:

3. Correct $\iff$ contracts to a single point

Implemented in linear time via union–find

[Danos 1990, Guerrini 1999]
\( P \perp Q \otimes P \& Q \)
\[(P \perp \otimes Q) \rightarrow (P \perp \otimes Q) \rightarrow P \& Q \]
\[ \overline{\mathcal{Q}^\perp, \mathcal{Q}} \]
\( \overline{p \perp}, \overline{p} \quad \overline{q \perp}, \overline{q} \)
\[ \overline{p^\perp, p} \quad \overline{q^\perp, q} \]
\[ P^\perp, P \quad \Rightarrow \quad Q^\perp, Q \]

\[ (P^\perp \otimes Q^\perp), P, Q \]
\[ P^\perp, P \quad Q^\perp, Q \]

\[ (P^\perp \otimes Q^\perp), P, Q \]
\[ \frac{P^\perp, P}{Q^\perp, Q} \frac{(P^\perp \otimes Q^\perp), P, Q}{(P^\perp \otimes Q^\perp), P \otimes Q} \]
\[
\begin{align*}
\quad & P^\perp, P \quad Q^\perp, Q \\
\quad & (P^\perp \otimes Q^\perp), P, Q \\
\quad & (P^\perp \otimes Q^\perp), P \& Q
\end{align*}
\]
\[
\frac{P^\perp, P}{(P^\perp \otimes Q^\perp), P, Q}
\]
\[
\frac{Q^\perp, Q}{(P^\perp \otimes Q^\perp), P \& Q}
\]
\[
(P^\perp \otimes Q^\perp) \& P \& Q
\]
\[
\begin{array}{c}
P^\perp, P \quad Q^\perp, Q \\
(P^\perp \otimes Q^\perp), P, Q \\
(P^\perp \otimes Q^\perp), P \& Q \\
(P^\perp \otimes Q^\perp) \& P \& Q
\end{array}
\]
For multiplicative linear logic:

- We have contractibility as abstract sequentialization
- Proof net correctness is linear-time decidable
For **multiplicative** linear logic:

- We have **contractibility** as abstract **sequentialization**
- Proof net **correctness** is **linear-time** decidable

What about **additive** linear logic?
Additive proof nets

(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)

\[
A + (B \times C) = (A + B) \times (A + C)
\]

Correctness

- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]
Additive proof nets

(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)

\[
A + (B \times C) = (A + B) \times (A + C)
\]

Correctness

- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]
Additive proof nets
(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)

$A + (B \times C) = (A + B) \times (A + C)$

Correctness
- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]
Additive proof nets

(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)

\[
\frac{A + (B \times C)}{(A + B) \times (A + C)}
\]

Correctness

- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]
Additive proof nets

(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)

$A + (B \times C)$

$(A + B) \times (A + C)$

Correctness

- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]
Additive proof nets

(We use $A + B$ and $A \times B$ instead of $A \oplus B$ and $A \& B$)

\[
A + (B \times C) = (A + B) \times (A + C)
\]

Correctness

- A resolution chooses one branch of each $+$ above and $\times$ below
- Every resolution must contain exactly one link

[DH & Van Glabbeek 2005]
Composition is path composition

\[ \begin{align*}
P \times Q \\
Q \times P \\
P \times Q
\end{align*} \quad \rightarrow \quad \begin{align*}
P \times Q
\end{align*} \]
Additive linear logic / sum–product logic

Formulae:

\[ A, B, C ::= P \mid A + B \mid A \times B \]

Sequents: \( A \vdash B \) (or \( \vdash A^\perp, B \) where \( (\cdot)^\perp \) is DeMorgan dualization)

Sequent calculus:

\[
\begin{align*}
A \vdash B_i & \quad & A \vdash B & A \vdash C \\
\frac{A \vdash B_0 + B_1}{A \vdash B_0 + B_1} & & \frac{A \vdash B \quad A \vdash C}{A \vdash B \times C} \\
\frac{P \vdash P}{A \vdash P} & & \frac{A_i \vdash B}{A_0 \times A_1 \vdash B}
\end{align*}
\]

\[ A \vdash C \quad B \vdash C \]

\[ A + B \vdash C \]
Additive proof nets

A linking for a sequent $A \vdash B$ is a set of links $C \rightarrow D$:

- $C$ is a subformula of $A$
- $D$ is a subformula of $B$
- $C \rightarrow D$ is an axiom link if $C$ and $D$ are the same atom $P$

Correctness

- A resolution deletes one child of each $+$ in $A$ and $\times$ in $B$
- A linking is discrete if every resolution contains one link

A proof net is a discrete axiom linking
\[\frac{A \vdash B_i}{A \vdash B_0 + B_1}\]

\[\frac{A \vdash C \quad B \vdash C}{A + B \vdash C}\]

\[\frac{A \vdash B \quad A \vdash C}{A \vdash B \times C}\]

\[\frac{A_i \vdash B}{A_0 \times A_1 \vdash B}\]
Petri nets

A model of concurrent computation

- A set $\mathcal{P}$ of places $\mathcal{O}$
- A set $\leftrightarrow$ of transitions $S \leftrightarrow T$ where $S, T \subseteq \mathcal{P}$
- A configuration, a set $M \subseteq \mathcal{P}$ of places that store a token $\bullet$

Firing: if $S \leftrightarrow T$ and $S \subseteq M$ then $M \rightarrow (M \setminus S) \cup T$

(Models where places store multiple tokens have multisets $S$, $T$, and $M$)
Encoding $+ \text{ and } \times$

The nets $\mathcal{N}(P + Q)$ and $\mathcal{N}(P \times Q)$
Encoding a proof net

The net $\mathcal{N}(A \vdash B)$ is the cartesian product $\mathcal{N}(A^\perp) \times \mathcal{N}(B)$

Places: $\mathcal{P} = \mathcal{P}_A \times \mathcal{P}_B$

Transitions:

$$\{p\} \times S_B \quad \Rightarrow \quad \{p\} \times T_B \quad (p \in \mathcal{P}_A, \ S_B \Rightarrow T_B)$$

$$S_A \times \{q\} \quad \Rightarrow \quad T_A \times \{q\} \quad (S_A \Rightarrow T_A, \ q \in \mathcal{P}_B)$$

Configuration: a token in $(p, q)$ for each axiom link $P \vdash Q$
Firing a proof net

The net is correct if a single token at the root remains.
Firing a proof net

The net is correct if a single token at the root remains.
Firing a proof net

The net is correct if a single token at the root remains
Firing a proof net

The net is correct if a single token at the root remains
Firing a proof net

The net is **correct** if a single token at the root remains
Coalescence

\[ \begin{align*}
A \times B & \rightarrow A \times B \\
\downarrow & \downarrow \\
C & C
\end{align*} \]

\[ \begin{align*}
A \times B & \leftarrow A \times B \\
\uparrow & \uparrow \\
C & C
\end{align*} \]

\[ \begin{align*}
A & \rightarrow A \\
\downarrow & \downarrow \\
B + C & B + C
\end{align*} \]

\[ \begin{align*}
A & \leftarrow A \\
\uparrow & \uparrow \\
B + C & B + C
\end{align*} \]

\[ \begin{align*}
A + B & \rightarrow A + B \\
\downarrow & \downarrow \\
C & C
\end{align*} \]

\[ \begin{align*}
A + B & \leftarrow A + B \\
\uparrow & \uparrow \\
C & C
\end{align*} \]
\[
\begin{aligned}
\quad & 
\end{aligned}
\]
Results

Theorem
Coalescence for $L : A \vdash B$ gives $\{A \rightarrow B\}$ if and only if $L$ is discrete
Results

Theorem
Coalescence for $L : A \vdash B$ gives $\{A \rightarrow B\}$ if and only if $L$ is discrete

Theorem
Correctness of an additive proof net $L : A \vdash B$ is decidable in time

$O(|A| \times |B|)$
Results

Theorem
Coalescence for $L : A \vdash B$ gives $\{A \rightarrow B\}$ if and only if $L$ is discrete.

Theorem
Correctness of an additive proof net $L : A \vdash B$ is decidable in time

$$O(|A| \times |B|)$$

Theorem
Correctness of an additive proof net $L : A \vdash B$ is decidable in time

$$O(|L| \times (dA + dB) \times \max(\log|A|, \log|B|))$$
For additive linear logic:

- We have coalescence as abstract sequentialization
- Proof net correctness is (almost) linear-time decidable
Remark

The set of subsets of a set $X$ ordered by inclusion ($\subseteq$)

- Is a **free distributive lattice**: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Models ALL:
  - $A \vdash B \Rightarrow A \subseteq B$
  - $A \times B \Rightarrow A \cap B$
  - $A + B \Rightarrow A \cup B$
- Correctness: every resolution contains **at least one link**

- Diagram:

```
  P ∩ Q  P ∩ Q  P  P  Q  P ∪ P
    \  /    \  /    \  /    \  /
   P  Q  P ∩ P  P ∪ Q  P ∪ Q  P ∪ Q  P
```
Remark

The set of subsets of a set \( X \) ordered by inclusion (\( \subseteq \))

- Is a free distributive lattice: \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)
- Models ALL: \( A \vdash B \Rightarrow A \subseteq B \)
  \( A \times B \Rightarrow A \cap B \)
  \( A + B \Rightarrow A \cup B \)
- Correctness: every resolution contains at least one link

\[
P \cap Q \quad P \cap Q \quad P \quad P \quad Q \quad P \cup P
\]

\[
P \quad Q \quad P \cap P \quad P \cup Q \quad P \cup Q \quad P
\]

But distributivity destroys coalescence:

\[
(A \cup B) \cap (A \cup C)
\]

\[
A \cup (B \cap C)
\]
Proof search
\[ A \times B \vdash C + D \]

- \[ A \vdash C + D \]
  - \[ A \vdash C \]
  - \[ A \vdash D \]
- \[ B \vdash C + D \]
  - \[ B \vdash C \]
  - \[ B \vdash D \]
- \[ A \times B \vdash C \]
  - \[ A \vdash C \]
  - \[ B \vdash C \]
- \[ A \times B \vdash D \]
  - \[ A \vdash D \]
  - \[ B \vdash D \]
$P \times (Q \times R)$

$(P \times Q) \times R$

$P \times (Q \times R)$

$(P \times Q) \times R$
\[
A \times B \\
\quad C
\]

\[
A \times B \\
\quad C
\]

\[
A + B \\
\quad C
\]

\[
A \\
B \times C
\]

\[
A \\
B + C
\]

\[
A \\
B + C
\]

\[
P \times (Q \times P)
\]

\[
(Q \times P) \times Q
\]
\[ A \times B \quad A \times B \quad A + B \quad A \quad A \quad A \]

\[ C 

\]

\[ B \times C \quad B + C \quad B + C \]

\[ P \times (Q \times P) 

\]

\[ (Q \times P) \times Q \]
\[
A \times B \quad \quad A \times B \quad \quad A + B \quad \quad A \quad \quad A \\
C \quad \quad C \quad \quad C \quad \quad B \times C \quad \quad B + C \quad \quad B + C
\]

\[
P \times (Q \times P) \\
(Q \times P) \times Q
\]
\[ A \times B \]
\[ A \times B \]
\[ A + B \]
\[ A \]
\[ A \]
\[ A \]
\[ A \]
\[ A \]
\[ A \]

\[ P \times (Q \times P) \]
\[ (Q \times P) \times Q \]
More results

- Proof search on $A \vdash B$ is $O(|A| \times |B|)$
More results

- **Proof search** on $A \vdash B$ is $O(|A| \times |B|)$

For additive linear logic with units:

- **Proof search** on $A \vdash B$ is $O(|A| \times |B|)$
- **Correctness** of $L : A \vdash B$ is $O(|A| \times |B|)$
More results

- **Proof search** on $A \vdash B$ is $O(|A| \times |B|)$

For additive linear logic with units:

- **Proof search** on $A \vdash B$ is $O(|A| \times |B|)$
- **Correctness** of $L : A \vdash B$ is $O(|A| \times |B|)$

For first-order additive linear logic:

- **Proof search** is NP-complete
Further work

- First-order proofs without syntax
- Coalescence for MALL-nets
- Proof search in classical logic