The atomic λ-calculus

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Sharing and Sequentiality in Proof Systems with Locality
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** PPS, Paris 7

The atomic λ-calculus

Typeable — Curry–Howard for deep inference

Atomic — Duplication of individual constructors

Sharing — Fully lazy sharing

PSN — Preservation of strong normalisation w.r.t λ-calculus

Open deduction

$$\begin{array}{c} A \\ \Downarrow \\ C \end{array} \qquad := \qquad a \qquad \frac{A}{C}\,r \qquad \begin{array}{c} A_1 \\ \Downarrow \\ C_1 \end{array} \wedge \begin{array}{c} A_2 \\ \Downarrow \\ C_2 \end{array} \qquad \begin{array}{c} C_1 \\ \Uparrow \\ A_1 \end{array} \to \begin{array}{c} A_2 \\ \Downarrow \\ C_2 \end{array} \qquad \begin{array}{c} A \\ \Downarrow \\ B \\ \Downarrow \\ C \end{array}$$

contraction $\dfrac{A}{A \wedge A}$ $\dfrac{A \vee A}{A}$ co-contraction

axiom $\dfrac{B}{A \rightarrow (B \wedge A)}$ $\dfrac{(A \rightarrow B) \wedge A}{B}$ cut / evaluation

medial

$$\frac{(A \vee B) \rightarrow (C \wedge D)}{(A \rightarrow C) \wedge (B \rightarrow D)}$$

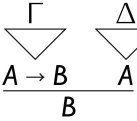
Natural deduction

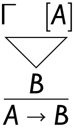
$$\dfrac{\begin{array}{c}\Gamma\\ \bigtriangledown\\ A \to B\end{array} \qquad \begin{array}{c}\Delta\\ \bigtriangledown\\ A\end{array}}{B} \qquad\qquad \dfrac{\begin{array}{c}\Gamma \quad [A]\\ \bigtriangledown\\ B\end{array}}{A \to B}$$

Open deduction

$$\dfrac{\begin{array}{ccc}\Gamma & & \Delta\\ \Downarrow & \wedge & \Downarrow\\ A \to B & & A\end{array}}{B}$$

Natural deduction



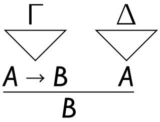


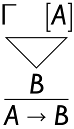
Open deduction

$$\cfrac{\begin{array}{c}\Gamma\\ \Downarrow\\ A\rightarrow B\end{array} \wedge \begin{array}{c}\Delta\\ \Downarrow\\ A\end{array}}{B}$$

$$\begin{array}{c}\Gamma\wedge A\wedge\cdots\wedge A\\ \Downarrow\\ B\end{array}$$

Natural deduction





Open deduction

$$\frac{\begin{array}{c}\Gamma\\ \Downarrow\\ A \to B\end{array} \quad \wedge \quad \begin{array}{c}\Delta\\ \Downarrow\\ A\end{array}}{B}$$

$$\begin{array}{c}\Gamma \wedge \dfrac{A}{A \wedge \cdots \wedge A}\\ \Downarrow\\ B\end{array}$$

Natural deduction

Open deduction

$$\dfrac{\begin{array}{c}\Gamma\\ \Downarrow\\ A\to B\end{array} \quad\wedge\quad \begin{array}{c}\Delta\\ \Downarrow\\ A\end{array}}{B}$$

$$\dfrac{\Gamma}{A \to \begin{array}{c}\Gamma \wedge \dfrac{A}{A\wedge\cdots\wedge A}\\ \Downarrow\\ B\end{array}}$$

Atomicity

$$\frac{(A \vee B) \to (C \wedge D)}{(A \to C) \wedge (B \to D)}$$

$$\frac{A \to B}{(A \to B) \wedge (A \to B)} \quad \rightsquigarrow \quad \frac{\dfrac{A}{A \vee A} \to \dfrac{B}{B \wedge B}}{(A \to B) \wedge (A \to B)}$$

$$\frac{(A \lor B) \to (C \land D)}{(A \to C) \land (B \to D)}$$

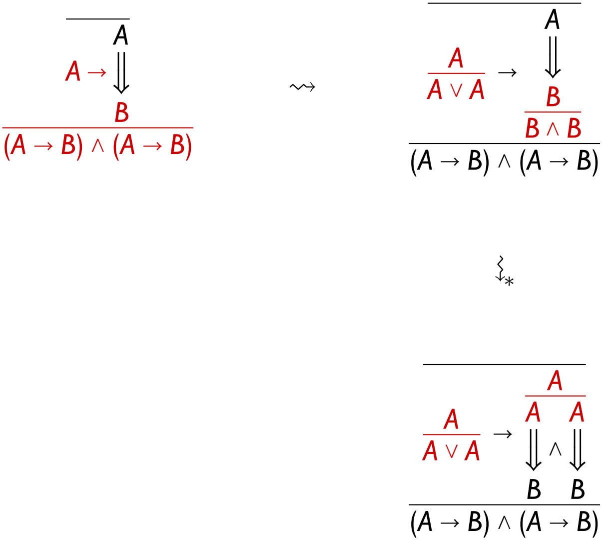
$$\frac{A \to B}{(A \to B) \land (A \to B)} \quad \rightsquigarrow \quad \frac{\dfrac{A}{A \lor A} \to \dfrac{B}{B \land B}}{(A \to B) \land (A \to B)}$$

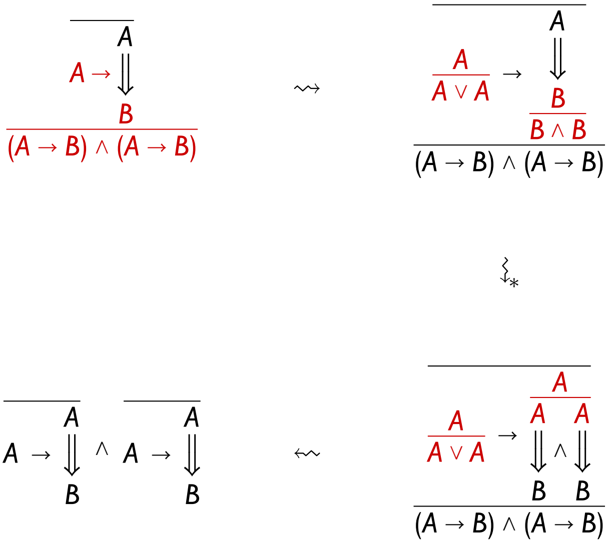
$$\frac{A \land B}{(A \land B) \land (A \land B)} \quad \rightsquigarrow \quad \frac{A}{A \land A} \land \frac{B}{B \land B}$$

$$\frac{A}{A \land A} \qquad \frac{A \lor A}{A} \qquad \frac{a}{a \land a} \qquad \frac{a \lor a}{a}$$

$$\frac{\begin{array}{r}\overline{A}\\ A \to \Downarrow\\ B\end{array}}{(A \to B) \wedge (A \to B)}$$

$$\dfrac{\begin{array}{r} \dfrac{}{\hphantom{A \to{}} A} \\ A \to \Big\Downarrow \\ B \end{array}}{(A \to B) \wedge (A \to B)} \quad \rightsquigarrow \quad \dfrac{\begin{array}{ccc} \multicolumn{3}{r}{\dfrac{}{\hphantom{A \vee A \to{}} A}} \\ \dfrac{A}{A \vee A} & \to & \Big\Downarrow \\ & & \dfrac{B}{B \wedge B} \end{array}}{(A \to B) \wedge (A \to B)}$$


A
A →
B
(A → B) ∧ (A → B)
⇝
A
A
A ∨ A
→
B
B ∧ B
(A → B) ∧ (A → B)
⇝*
A
A
A ∨ A
→
A
A
A
∧
B
B
(A → B) ∧ (A → B)


A
A →
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(A → B) ∧ (A → B)
A
A ∨ A
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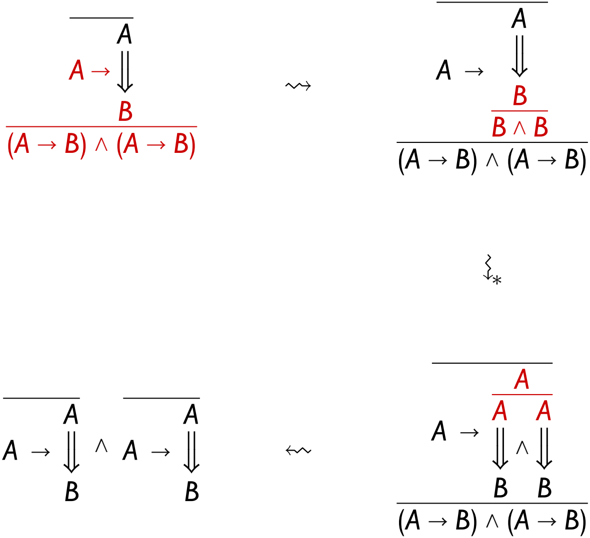
B ∧ B
(A → B) ∧ (A → B)
A
A ∨ A
A
A
A
→
∧
B
B
(A → B) ∧ (A → B)
A → A ∧ A → A
B
B

Distributor

$$\frac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)} \quad := \quad \frac{\dfrac{A}{A \vee A} \to (B \wedge C)}{(A \to B) \wedge (A \to C)}$$


A
A →
B
(A → B) ∧ (A → B)
A
A →
B
B ∧ B
(A → B) ∧ (A → B)
*
A
A A
A →
∧
B B
(A → B) ∧ (A → B)
A
A →
B
∧
A
A →
B

The atomic λ-calculus

$$A^{x}$$

x

$$\dfrac{\begin{array}{c}\Gamma\\ t\,\Big\Downarrow\\ A\to B\end{array}\quad\wedge\quad\begin{array}{c}\Delta\\ \Big\Downarrow\,u\\ A\end{array}}{B}$$

$(t)u$

$$\begin{array}{rc} & \dfrac{\Gamma}{\Gamma\wedge A^{x}}\\ A^{x}\to & \Big\Downarrow\,t\\ & B\end{array}$$

$\lambda x.t$

$$\begin{tikzcd}
\underbrace{A^{x_1} \wedge \ldots \wedge A^{x_n} \wedge \Delta} \arrow[d, Rightarrow, "u"] \\
B
\end{tikzcd}$$

u

$$\begin{array}{c} \underbrace{\dfrac{A}{A^{x_1} \wedge \ldots \wedge A^{x_n}} \wedge \Delta} \\ \Big\Downarrow u \\ B \end{array}$$

$$u\left[x_1, \ldots, x_n \leftarrow \quad \right]$$

$$\begin{array}{c}\underbrace{\dfrac{\begin{array}{c}\Gamma\\ t\,\Downarrow\\ A\end{array}}{A^{x_1}\wedge\ldots\wedge A^{x_n}}\wedge\Delta}\\ \Downarrow u\\ B\end{array}$$

$$u\left[x_1,\ldots,x_n \leftarrow t\right]$$

$$\begin{array}{c}\underbrace{\dfrac{\begin{array}{c}\Gamma\\ t\,\Downarrow\\ A\end{array}}{A^{x_1}\wedge\ldots\wedge A^{x_n}}\wedge\Delta}\\ \Downarrow u\\ B\end{array}$$

$$u\left[x_1,\ldots,x_n \leftarrow\ t\,\right]$$

Explicit substitution: $u[x := t]$

Functional programming: `let` x `be` t `in` u

$$\frac{A \to (B_1 \wedge \cdots \wedge B_n)}{(A \to B_1) \quad \wedge \cdots \wedge (A \to B_n)}$$

$$\frac{A \to (B_1 \wedge \cdots \wedge B_n)}{(A \to B_1) \quad \wedge \cdots \wedge (A \to B_n)}$$

$$\begin{tikzcd}
\Delta_1 \arrow[dd, Rightarrow, "t_1"'] & & \Delta_n \arrow[dd, Rightarrow, "t_n"] \\
& \wedge \cdots \wedge & \\
B_1 & & B_n
\end{tikzcd}$$

$$t^n := \langle t_1, \ldots, t_n \rangle$$

$$\frac{A \to (B_1 \wedge \cdots \wedge B_n)}{(A \to B_1) \quad \wedge \cdots \wedge (A \to B_n)}$$

$$\begin{tikzcd}
 & \Gamma \arrow[d, Rightarrow, "{[c_1]\cdots[c_n]}"] & \\
 & \overbrace{\hspace{8em}} & \\
\Delta_1 \arrow[d, Rightarrow, "t_1"'] & \wedge \cdots \wedge & \Delta_n \arrow[d, Rightarrow, "t_n"] \\
B_1 & & B_n
\end{tikzcd}$$

$$t^n := \langle t_1, \ldots, t_n \rangle \mid t^n[c]$$

$$\frac{A \to \begin{array}{c} \Gamma \wedge A \\ \Big\Downarrow \, t^n \\ B_1 \wedge \cdots \wedge B_n \end{array}}{(A \to B_1) \quad \wedge \cdots \wedge (A \to B_n)}$$

$$\begin{array}{ccc} & \Gamma & \\ & \Big\Downarrow \, [c_1] \cdots [c_n] & \\ \multicolumn{3}{c}{\overbrace{\hspace{8em}}} \\ \Delta_1 & & \Delta_n \\ t_1 \Big\Downarrow & \wedge \cdots \wedge & \Big\Downarrow t_n \\ B_1 & & B_n \end{array}$$

t^n

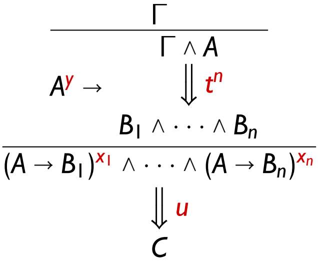
$t^n := \langle t_1, \ldots, t_n \rangle \mid t^n[c]$

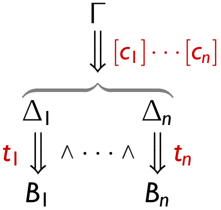
$$\begin{array}{c} \Gamma \\ \hline \Gamma \wedge A \\ A^{y} \to \qquad \Big\Downarrow\, t^{n} \\ B_1 \wedge \cdots \wedge B_n \\ \hline (A \to B_1) \quad \wedge \cdots \wedge (A \to B_n) \end{array}$$

$$\begin{tikzcd} & \Gamma \arrow[d, Rightarrow, "{[c_1]\cdots[c_n]}"] & \\ & \overbrace{\hspace{4cm}} & \\ \Delta_1 \arrow[d, Rightarrow, "t_1"'] & \wedge \cdots \wedge & \Delta_n \arrow[d, Rightarrow, "t_n"] \\ B_1 & & B_n \end{tikzcd}$$

$$\lambda y.t^{n}$$

$$t^{n} := \langle t_1, \ldots, t_n \rangle \mid t^{n}[c]$$





$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.t^n]$

$t^n := \langle t_1, \ldots, t_n \rangle \mid t^n[c]$

Terms $t, u, v \quad := \quad x \quad | \quad \lambda x.t \quad | \quad (t)u \quad | \quad t[c]$

Closures $[c], [d] \quad := \quad [x_1, \ldots, x_n \leftarrow t] \quad | \quad [x_1, \ldots, x_n \twoheadleftarrow \lambda y.t^n]$

n-Terms $t^n \quad := \quad \langle t_1, \ldots, t_n \rangle \quad | \quad t^n[c]$

Denotation

$$
\begin{array}{lclclcl}
N, M & := & x & \mid & (N)M & \mid & \lambda x.N \\
t, u & := & x_i & \mid & (t)u & \mid & \lambda x.t[x_1, \ldots, x_n \leftarrow x]
\end{array}
$$

Notation

Closure sequence	$[C]\ ;\ [c_i]_{i \leq n}$	$:=$	$[c_1] \ldots [c_n]$
Vector notation	$\vec{x}\ ;\ \vec{t}$	$:=$	$x_1, \ldots, x_n\ ;\ t_1, \ldots, t_n$
Substitution (linear)	$\{t/x\}$		
Substitution sequence	$\{t_i/x_i\}_{i \leq n}$	$:=$	$\{t_1/x_1\} \ldots \{t_n/x_n\}$

$$\llbracket x \rrbracket = x \qquad\qquad \llbracket \lambda x.t \rrbracket = \lambda x.\llbracket t \rrbracket$$

$$\llbracket (t)u \rrbracket = (\llbracket t \rrbracket)\llbracket u \rrbracket \qquad\qquad \llbracket t\,[c] \rrbracket = \llbracket t \rrbracket \,\{\!|c|\!\}$$

$$\{\!|x_1, \ldots, x_n \leftarrow t|\!\} = \{\llbracket t \rrbracket / x_i\}_{i \leq n}$$

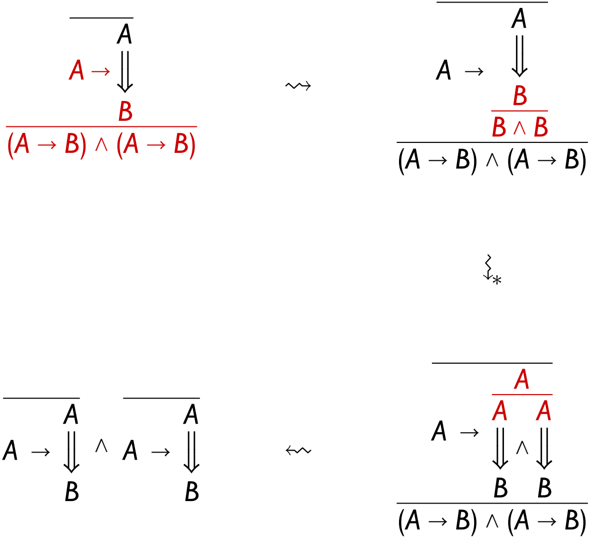
$$\llbracket x \rrbracket = x \qquad \llbracket \lambda x.t \rrbracket = \lambda x.\llbracket t \rrbracket$$

$$\llbracket (t)u \rrbracket = (\llbracket t \rrbracket)\llbracket u \rrbracket \qquad \llbracket t\,[c] \rrbracket = \llbracket t \rrbracket\,\lBrace c \rBrace$$

$$\lBrace x_1, \ldots, x_n \leftarrow t \rBrace = \{\llbracket t \rrbracket / x_i\}_{i \leq n}$$

$$\lBrace x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle t_1, \ldots, t_n \rangle [C] \rBrace = \{\lambda y.\llbracket t_i \rrbracket\,\lBrace C \rBrace / x_i\}_{i \leq n}$$

Normalisation



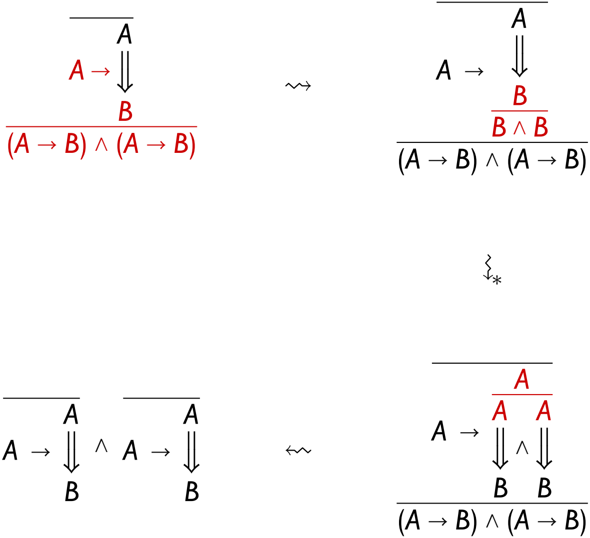
A → A ⇓ B
(A → B) ∧ (A → B)
⇝
A → A ⇓ B
B ∧ B
(A → B) ∧ (A → B)
⇝*
A → A A A ⇓ ∧ ⇓ B B
(A → B) ∧ (A → B)
⇜
A → A ⇓ B ∧ A → A ⇓ B

$$\cfrac{\begin{array}{r} \overline{A} \\ A \to \Downarrow \\ B \end{array}}{(A \to B) \wedge (A \to B)} \qquad \rightsquigarrow \qquad \cfrac{\begin{array}{r} \overline{A} \\ A \to \quad \Downarrow \\ \cfrac{B}{B \wedge B} \end{array}}{(A \to B) \wedge (A \to B)}$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t]$$

$$\rightsquigarrow_s$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n \rangle [z_1, \ldots, z_n \leftarrow t]]$$


A
$A \rightarrow \Downarrow$
B
$(A \rightarrow B) \wedge (A \rightarrow B)$
$\rightsquigarrow$
A
$A \rightarrow \Downarrow$
B
$B \wedge B$
$(A \rightarrow B) \wedge (A \rightarrow B)$
$\rightsquigarrow^*$
A
$A \quad A$
$A \rightarrow \Downarrow \wedge \Downarrow$
$B \quad B$
$(A \rightarrow B) \wedge (A \rightarrow B)$
$\leftsquigarrow$
A
$A \rightarrow \Downarrow$
B
$\wedge$
A
$A \rightarrow \Downarrow$
B

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle t_1, \ldots, t_n\rangle[\vec{z}_1, \ldots, \vec{z}_n \leftarrow y]]$$

$$\rightsquigarrow_s$$

$$u\{\lambda y_i.t_i[\vec{z}_i \leftarrow y_i]/x_i\}_{i\leq n}$$

$$\overline{A \to \begin{array}{c} A \\ \Downarrow \\ B \end{array}} \wedge \overline{A \to \begin{array}{c} A \\ \Downarrow \\ B \end{array}} \qquad \leftsquigarrow \qquad \dfrac{\overline{A \to \begin{array}{cc} \multicolumn{2}{c}{A} \\ \hline A & A \\ \Downarrow \wedge & \Downarrow \\ B & B \end{array}}}{(A \to B) \wedge (A \to B)}$$

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \Downarrow\\ A\to B\end{matrix}\quad\wedge\quad\begin{matrix}\Delta\\ \Downarrow\\ A\end{matrix}}{B}}{B\wedge B}\quad\rightsquigarrow\quad\begin{array}{c}\dfrac{\begin{matrix}\Gamma\\ \Downarrow\\ A\to B\end{matrix}}{(A\to B)\wedge(A\to B)}\quad\wedge\quad\dfrac{\begin{matrix}\Delta\\ \Downarrow\\ A\end{matrix}}{A\wedge A}\\ \hline\hline \dfrac{(A\to B)\wedge A}{B}\quad\wedge\quad\dfrac{(A\to B)\wedge A}{B}\end{array}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \Downarrow\\ A\to B\end{array} \quad\wedge\quad \begin{array}{c}\Delta\\ \Downarrow\\ A\end{array}}{\dfrac{B}{B\wedge B}} \quad\rightsquigarrow\quad \begin{array}{c} \begin{array}{c}\Gamma\\ \Downarrow\\ \dfrac{A\to B}{(A\to B)\wedge(A\to B)}\end{array} \quad\wedge\quad \begin{array}{c}\Delta\\ \Downarrow\\ \dfrac{A}{A\wedge A}\end{array} \\ \hline\hline \dfrac{(A\to B)\wedge A}{B} \quad\wedge\quad \dfrac{(A\to B)\wedge A}{B} \end{array}$$

$$t\,[x_1,\ldots,x_n \leftarrow (u)v]$$

$$\rightsquigarrow_s$$

$$t\,\{(y_i)z_i/x_i\}_{i\leq n}\,[y_1,\ldots,y_n \leftarrow u]\,[z_1,\ldots,z_n \leftarrow v]$$

$$t[c][d] \quad \sim \quad t[d][c]$$

$$\begin{array}{c}
\underbrace{\begin{array}{ccc}
\Gamma & & \Delta \\
{[c]}\ \Big\Downarrow & \wedge & \Big\Downarrow\ {[d]} \\
A & & B
\end{array}} \\
\Big\Downarrow\ t \\
C
\end{array}$$

if $[d]$ does not bind in $[c]$

$$t[c][d] \quad \sim \quad t[d][c]$$

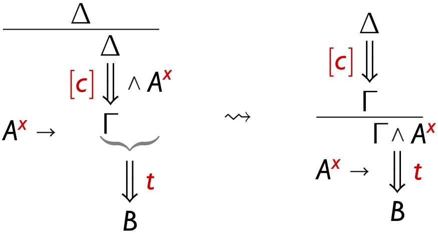
$$(\lambda x.t)u \quad \rightsquigarrow_{\beta} \quad t\{u/x\}$$

$$\dfrac{\dfrac{\Gamma}{A^{x} \to \begin{array}{c}\Gamma \wedge A^{x} \\ \Downarrow t \\ B\end{array}} \wedge \begin{array}{c}\Delta \\ \Downarrow u \\ A\end{array}}{B} \quad \rightsquigarrow \quad \begin{array}{c}\underbrace{\Gamma \wedge \begin{array}{c}\Delta \\ \Downarrow u \\ A\end{array}} \\ \Downarrow t \\ B\end{array}$$

$$t[c][d] \quad \sim \quad t[d][c]$$

$$(\lambda x.t)u \quad \rightsquigarrow_{\beta} \quad t\{u/x\}$$

$$\lambda x.t[c] \quad \rightsquigarrow_{s} \quad (\lambda x.t)[c]$$



if x is not free in $[c]$

$$\begin{array}{rcl}
t[c][d] & \sim & t[d][c] \\
\\
(\lambda x.t)u & \rightsquigarrow_{\beta} & t\{u/x\} \\
\\
\lambda x.t[c] & \rightsquigarrow_{s} & (\lambda x.t)[c] \\
(t[c])u & \rightsquigarrow_{s} & ((t)u)[c] \\
(t)u[d] & \rightsquigarrow_{s} & ((t)u)[d]
\end{array}$$

$$\begin{tikzcd}
\Gamma' \arrow[d, Rightarrow, "{[c]}"'] & & \Delta' \arrow[d, Rightarrow, "{[d]}"] \\
\Gamma \arrow[d, Rightarrow, "t"'] & \wedge & \Delta \arrow[d, Rightarrow, "u"] \\
\underline{A \to B} & \underline{\qquad} & \underline{A} \\
& B &
\end{tikzcd}$$

$$t[c][d] \quad \sim \quad t[d][c]$$

$$(\lambda x.t)u \quad \rightsquigarrow_\beta \quad t\{u/x\}$$

$$\begin{array}{rcl}
\lambda x.t[c] & \rightsquigarrow_s & (\lambda x.t)[c] \\
(t[c])u & \rightsquigarrow_s & ((t)u)[c] \\
(t)u[d] & \rightsquigarrow_s & ((t)u)[d] \\
u[\vec{x} \leftarrow t[c]] & \rightsquigarrow_s & u[\vec{x} \leftarrow t][c]
\end{array}$$

$$\begin{tikzcd}
\Delta \arrow[d, Rightarrow, "{[c]}"] \\
\Gamma \arrow[d, Rightarrow, "t"] \\
A^{x_1} \wedge \cdots \wedge A^{x_n}
\end{tikzcd}$$

$$t[c][d] \quad \sim \quad t[d][c]$$

$$(\lambda x.t)u \quad \rightsquigarrow_{\beta} \quad t\{u/x\}$$

$$\begin{array}{rcl}
\lambda x.t[c] & \rightsquigarrow_{s} & (\lambda x.t)[c] \\
(t[c])u & \rightsquigarrow_{s} & ((t)u)[c] \\
(t)u[d] & \rightsquigarrow_{s} & ((t)u)[d] \\
u[\vec{x} \leftarrow t[c]] & \rightsquigarrow_{s} & u[\vec{x} \leftarrow t][c] \\
u[\vec{x} \twoheadleftarrow \lambda y.t^{n}[c]] & \rightsquigarrow_{s} & u[\vec{x} \twoheadleftarrow \lambda y.t^{n}][c]
\end{array}$$

if y is not free in $[c]$

$$t[c][d] \quad \sim \quad t[d][c]$$

$$(\lambda x.t)u \quad \rightsquigarrow_\beta \quad t\{u/x\}$$

$$\begin{aligned}
\lambda x.t[c] &\quad \rightsquigarrow_s \quad (\lambda x.t)[c] \\
(t[c])u &\quad \rightsquigarrow_s \quad ((t)u)[c] \\
(t)u[d] &\quad \rightsquigarrow_s \quad ((t)u)[d] \\
u[\vec{x} \leftarrow t[c]] &\quad \rightsquigarrow_s \quad u[\vec{x} \leftarrow t][c] \\
u[\vec{x} \twoheadleftarrow \lambda y.t^n[c]] &\quad \rightsquigarrow_s \quad u[\vec{x} \twoheadleftarrow \lambda y.t^n][c]
\end{aligned}$$

$$u[\vec{y} \leftarrow y][\vec{x}, y, \vec{z} \leftarrow t] \quad \rightsquigarrow_s \quad u[\vec{x}, \vec{y}, \vec{z} \leftarrow t]$$

$$\frac{A}{A^{x_1} \wedge \cdots \wedge \dfrac{A^{y}}{A^{y_1} \wedge \cdots \wedge A^{y_n}} \wedge \cdots \wedge A^{z_k}} \quad \rightsquigarrow \quad \frac{A}{A^{x_1} \wedge \cdots \wedge A^{z_k}}$$

$$t[c][d] \quad \sim \quad t[d][c]$$

$$(\lambda x.t)u \quad \rightsquigarrow_\beta \quad t\{u/x\}$$

$$\begin{array}{rcl}
\lambda x.t[c] & \rightsquigarrow_s & (\lambda x.t)[c] \\
(t[c])u & \rightsquigarrow_s & ((t)u)[c] \\
(t)u[d] & \rightsquigarrow_s & ((t)u)[d] \\
u[\vec{x} \leftarrow t[c]] & \rightsquigarrow_s & u[\vec{x} \leftarrow t][c] \\
u[\vec{x} \twoheadleftarrow \lambda y.t^n[c]] & \rightsquigarrow_s & u[\vec{x} \twoheadleftarrow \lambda y.t^n][c]
\end{array}$$

$$u[\vec{y} \leftarrow y][\vec{x}, y, \vec{z} \leftarrow t] \quad \rightsquigarrow_s \quad u[\vec{x}, \vec{y}, \vec{z} \leftarrow t]$$

$$\begin{array}{rcl}
t[\vec{x} \leftarrow (u)v] & \rightsquigarrow_s & t\{(y_i)z_i/x_i\}_{i \leq n}[\vec{y} \leftarrow u][\vec{z} \leftarrow v] \\
u[\vec{x} \leftarrow \lambda y.t] & \rightsquigarrow_s & u[\vec{x} \twoheadleftarrow \lambda y.\langle \vec{z} \rangle[\vec{z} \leftarrow t]] \\
u[\vec{x} \twoheadleftarrow \lambda y.\langle \vec{t} \rangle[\vec{z} \leftarrow y]] & \rightsquigarrow_s & u\{\lambda y_i.t_i[\vec{z}_i \leftarrow y_i]/x_i\}_{i \leq n}
\end{array}$$

Example

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y][z_1, \ldots, z_n \leftarrow t]]$$

$$u[x_1,\ldots,x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1,\ldots,x_n \twoheadleftarrow \lambda y.\langle z_1,\ldots,z_n\rangle[z_1,\ldots,z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1,\ldots,x_n \twoheadleftarrow \lambda y.\langle z_1,\ldots,z_n\rangle[z_1,\ldots,z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1,\ldots,x_n \twoheadleftarrow \lambda y.\langle z_1,\ldots,z_n\rangle[\leftarrow y][z_1,\ldots,z_n \leftarrow t]]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y][z_1, \ldots, z_n \leftarrow t]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y]][z_1, \ldots, z_n \leftarrow t]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y][z_1, \ldots, z_n \leftarrow t]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y]][z_1, \ldots, z_n \leftarrow t]$$

$$u[x_1, \ldots, x_n \leftarrow \lambda y.t[\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t[\leftarrow y]]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[z_1, \ldots, z_n \leftarrow t][\leftarrow y]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y][z_1, \ldots, z_n \leftarrow t]]$$

$$u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n\rangle[\leftarrow y]][z_1, \ldots, z_n \leftarrow t]$$

$$u\,\{\lambda y_i.z_i[\leftarrow y_i]/x_i\}_{i \leq n}[z_1, \ldots, z_n \leftarrow t]$$

Full laziness

Skeleton of $\lambda y.t$: smallest $\lambda y.s$ such that

$$\lambda y.t = (\lambda y.s)\{u_i/x_i\}_{i\leq n}$$

(with non-capturing substitution)

u_i: maximal free subexpression (MFE)

Full laziness: skeletons are duplicated; MFE's remain shared

$$
\begin{aligned}
& u[x_1, \ldots, x_n \leftarrow \lambda y.t] \\
\leadsto \quad & u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle z_1, \ldots, z_n \rangle [z_1, \ldots, z_n \leftarrow t]] \\
\leadsto^{*} \quad & u[x_1, \ldots, x_n \twoheadleftarrow \lambda y.\langle s_1, \ldots, s_n \rangle [\vec{y}_1, \ldots, \vec{y}_n \leftarrow y]][C] \\
\leadsto \quad & u\{\lambda y_i.s_i[\vec{y}_i \leftarrow y]_i / x_i\}_{i \leq n}[C]
\end{aligned}
$$

Results

Sharing reduction ($\rightsquigarrow_s$)

- SN and confluent
- normal forms are the image of the λ-calculus

Normalisation ($\rightsquigarrow_s \cup \rightsquigarrow_\beta$)

- preserves SN w.r.t. λ-calculus (PSN)
- simple types give SN (by abstract reducibility)
- fully lazy sharing

The End