On MALL proof nets

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University of Bath

LL2016, Lyon

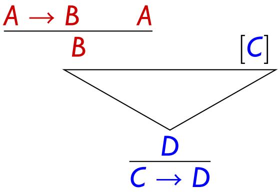
Intuitionistic logic	natural deduction / lambda-calculus
Linear logic	proof nets ?

Intuitionistic logic natural deduction / lambda-calculus

Linear logic proof nets ?

$$\dfrac{\vdash A \qquad \dfrac{B, C \vdash D}{B \vdash C \rightarrow D}}{A \rightarrow B \vdash C \rightarrow D}$$

$$\dfrac{\dfrac{\vdash A \qquad B, C \vdash D}{A \rightarrow B, C \vdash D}}{A \rightarrow B \vdash C \rightarrow D}$$



Intuitionistic logic — natural deduction / lambda-calculus

Linear logic — proof nets ?

$$\dfrac{\vdash A \qquad \dfrac{B, C \vdash D}{B \vdash C \to D}}{A \to B \vdash C \to D}$$

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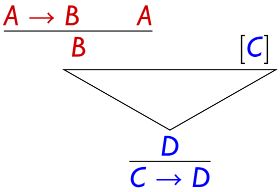
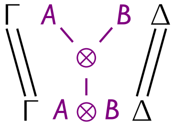


Motivation:
- Computation
- Canonicity

MLL proof nets

$$\frac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B, \Delta}$$



MLL proof nets

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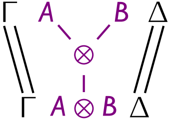
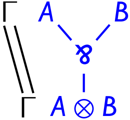
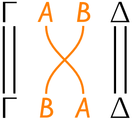


$$\overline{A, \bar{A}}$$

$$\frac{\Gamma, A, B}{\Gamma, A ⅋ B}$$



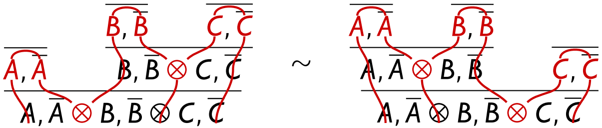


$$\frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta}$$

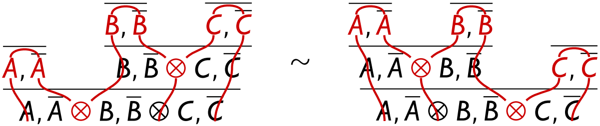
MLL proof nets

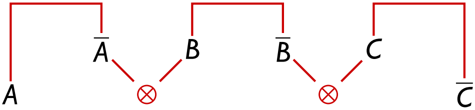
$$\dfrac{\dfrac{}{A,\overline{A}} \qquad \dfrac{\dfrac{}{B,\overline{B}} \qquad \dfrac{}{C,\overline{C}}}{B,\overline{B}\otimes C,\overline{C}}}{A,\overline{A}\otimes B,\overline{B}\otimes C,\overline{C}} \quad \sim \quad \dfrac{\dfrac{\dfrac{}{A,\overline{A}} \qquad \dfrac{}{B,\overline{B}}}{A,\overline{A}\otimes B,\overline{B}} \qquad \dfrac{}{C,\overline{C}}}{A,\overline{A}\otimes B,\overline{B}\otimes C,\overline{C}}$$

MLL proof nets

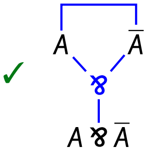


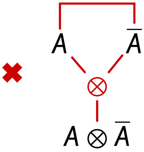
MLL proof nets



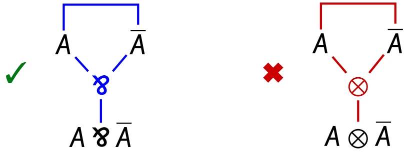


Correctness conditions





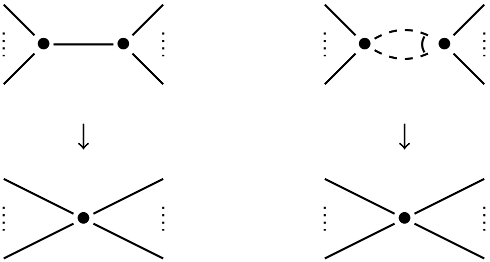
Correctness conditions



- long-trip [Girard 1987]
- switching [Danos & Regnier 1989]
- contractibility [Danos 1990]

MLL contractibility

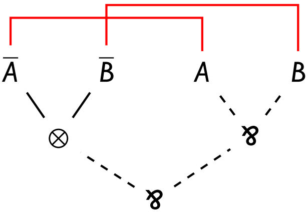
1. Start from an unlabelled graph with paired ⅋-edges
2. Contract by:

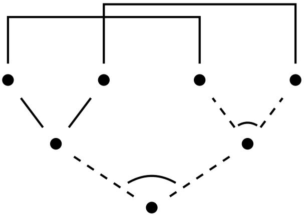


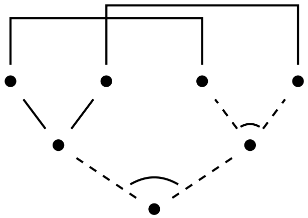
3. Correct $\Leftrightarrow$ contracts to a single point

Implemented in linear time via union–find

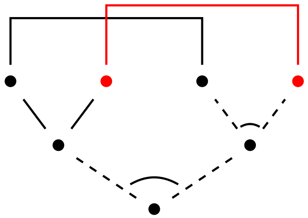
[Danos 1990, Guerrini 1999]


$\overline{A}$
$\overline{B}$
A
B
$\otimes$
⅋
⅋





$\overline{\overline{B}, B}$



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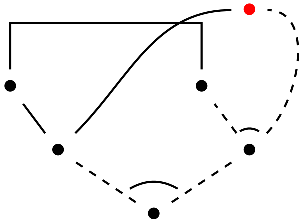
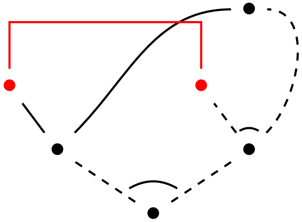


$\overline{\overline{A}, A}$ $\overline{\overline{B}, B}$



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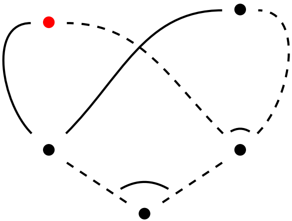
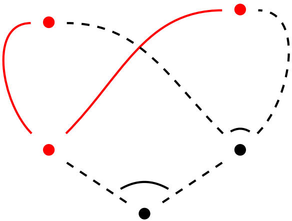


$$\frac{\overline{\bar{A}, A} \qquad \overline{\bar{B}, B}}{(\bar{A} \otimes \bar{B}), A, B}$$



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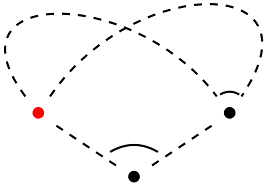
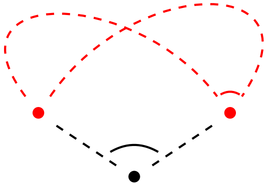


$$\dfrac{\dfrac{\dfrac{}{\overline{A}, A} \qquad \dfrac{}{\overline{B}, B}}{(\overline{A} \otimes \overline{B}), A, B}}{(\overline{A} \otimes \overline{B}), A \parr B}$$



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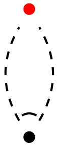
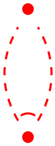


$$\dfrac{\dfrac{\dfrac{\dfrac{}{\overline{A}, A} \qquad \dfrac{}{\overline{B}, B}}{(\overline{A} \otimes \overline{B}), A, B}}{(\overline{A} \otimes \overline{B}), A \parr B}}{(\overline{A} \otimes \overline{B}) \parr A \parr B}$$



$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\overline{A}, A} \qquad \dfrac{}{\overline{B}, B}}{(\overline{A} \otimes \overline{B}), A, B}}{(\overline{A} \otimes \overline{B}), A \parr B}}{(\overline{A} \otimes \overline{B}) \parr A \parr B}
$$

Properties of MLL proof nets

De-sequentialization	$P \Longrightarrow N$	✓ linear-time
Sequentialization / correctness	$N \Longrightarrow P$	✓ linear-time
Composition / cut-elimination	$N_1 \circ N_2$	✓ P-time
Proof equivalence / canonicity	$\begin{array}{ccc} P_1 & \overset{?}{\sim} & P_2 \\ \Downarrow & & \Downarrow \\ N_1 & = & N_2 \end{array}$	✓

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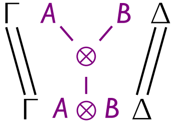
What about other fragments of linear logic?

Two traditions

$$\frac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B, \Delta}$$



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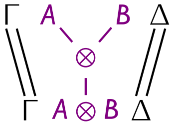
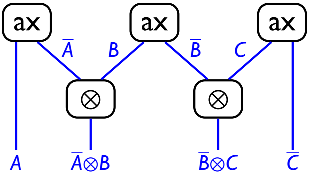


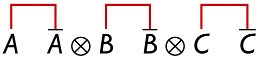
Computation

graph (nodes are rules)

Canonicity

sequent + axiom linking


ax
ax
ax
Ā
B
B̄
C
⊗
⊗
A
Ā⊗B
B̄⊗C
C̄


A Ā ⊗ B B̄ ⊗ C C̄

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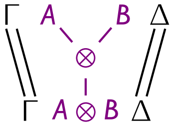

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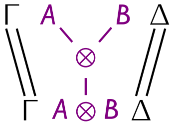
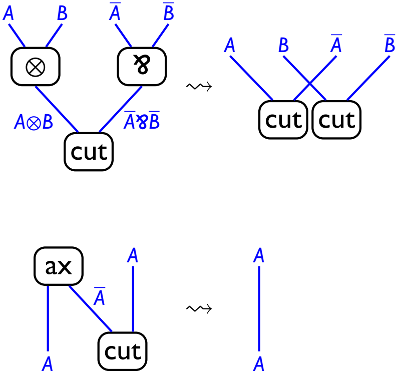


Computation	**Canonicity**
graph (nodes are rules)	sequent + axiom linking
normalization: graph rewriting	normalization: path composition



cut
cut
cut
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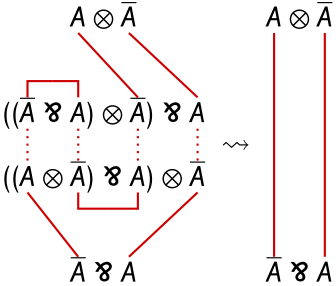
cut



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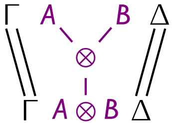

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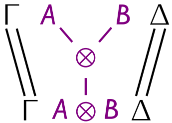
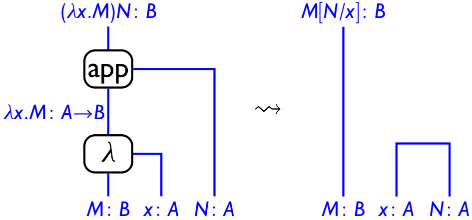
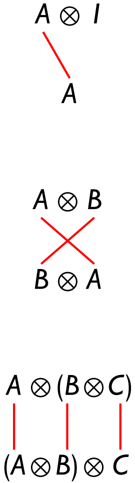


Computation	**Canonicity**
graph (nodes are rules)	sequent + axiom linking
normalization: graph rewriting	normalization: path composition
like lambda-calculus	like categorical coherence


(λx.M)N: B
app
λx.M: A→B
λ
M: B x: A N: A
⇝
M[N/x]: B
M: B x: A N: A


A ⊗ I
A
A ⊗ B
B ⊗ A
A ⊗ (B ⊗ C)
(A ⊗ B) ⊗ C

Two traditions

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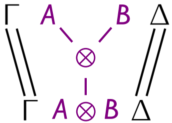

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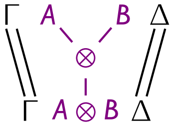

Computation	**Canonicity**
graph (nodes are rules)	sequent + axiom linking
normalization: graph rewriting	normalization: path composition
like lambda-calculus	like categorical coherence
good with exponentials	good with additives

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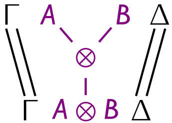

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graph (nodes are rules)	sequent + axiom linking
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good with exponentials	good with additives
interaction nets[1], sharing graphs[2]	??

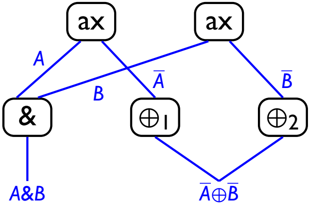
[1] [Lafont 1990] [2] [Lamping 1990, Asperti & Guerrini 1998]

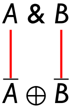
Additives

$$\dfrac{\dfrac{\dfrac{}{A\,,\;\overline{A}}}{A\,,\;\overline{A}\oplus\overline{B}} \qquad \dfrac{\dfrac{}{B\,,\;\overline{B}}}{B\,,\;\overline{A}\oplus\overline{B}}}{A \,\&\, B\,,\;\overline{A}\oplus\overline{B}}$$

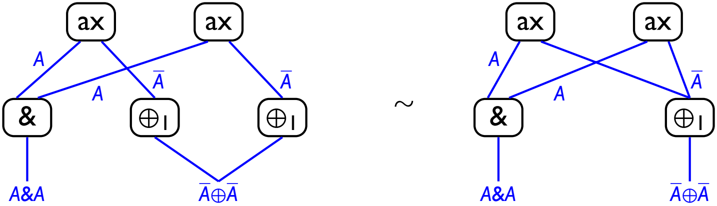
Additives

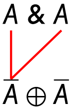
$$\dfrac{\dfrac{\overline{A\ ,\ \overline{A}}}{A\ ,\ \overline{A}\oplus\overline{B}} \qquad \dfrac{\overline{B\ ,\ \overline{B}}}{B\ ,\ \overline{A}\oplus\overline{B}}}{A\ \&\ B\ ,\ \overline{A}\oplus\overline{B}}$$



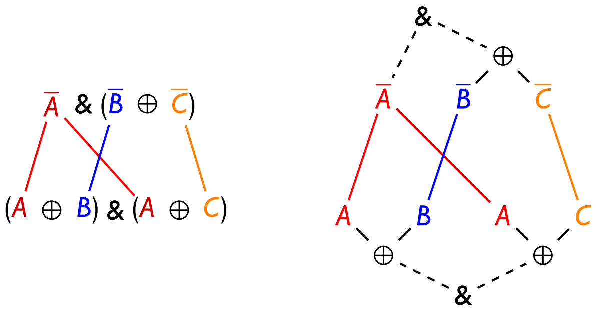


$$\dfrac{\dfrac{\overline{A\ ,\ \overline{A}}}{A\ ,\ \overline{A}\oplus\overline{A}}\ \oplus\mathsf{l} \qquad \dfrac{\overline{A\ ,\ \overline{A}}}{A\ ,\ \overline{A}\oplus\overline{A}}\ \oplus\mathsf{l}}{A \mathbin{\&} A\ ,\ \overline{A}\oplus\overline{A}} \quad\sim\quad \dfrac{\dfrac{\overline{A\ ,\ \overline{A}} \qquad \overline{A\ ,\ \overline{A}}}{A \mathbin{\&} A\ ,\ \overline{A}}\ \oplus\mathsf{l}}{A \mathbin{\&} A\ ,\ \overline{A}\oplus\overline{A}}$$





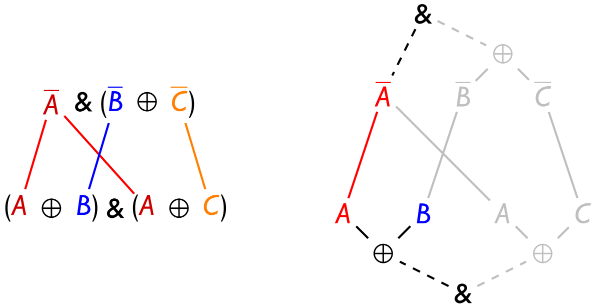
Additive proof nets



Correctness

- A resolution or slice deletes one child of each &
- Every resolution must contain exactly one link

Additive proof nets



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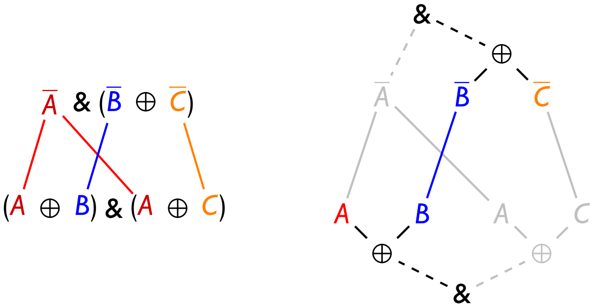

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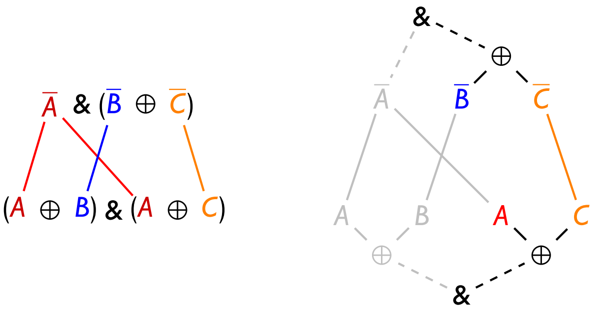

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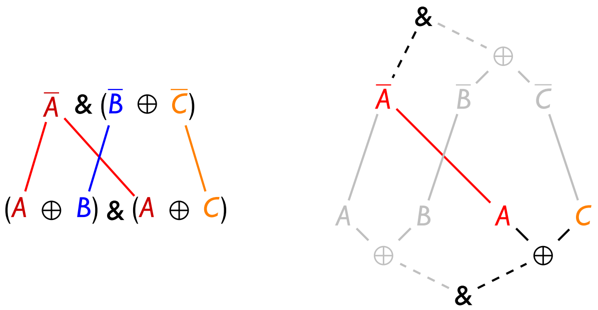

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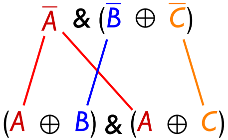

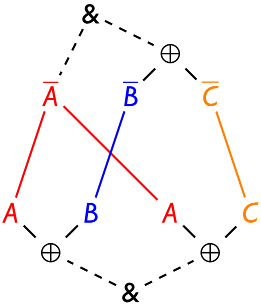

Correctness

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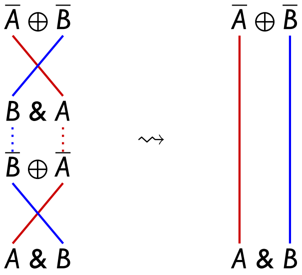
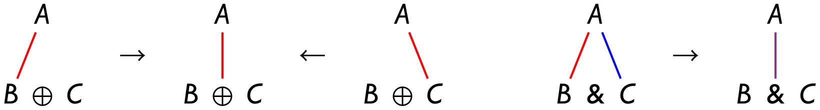
- Every resolution must contain exactly one link

Composition is path composition



ALL: Coalescence

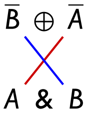


Correct ⇔ contracts to a single link between roots

[H & Hughes 2015]

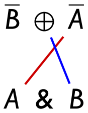
ALL: Coalescence

$\overline{\overline{A}\ ,\ A}$ $\qquad$ $\overline{\overline{B}\ ,\ B}$



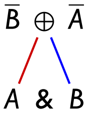
ALL: Coalescence

$$\overline{\;\bar{A}\ ,\ A\;} \qquad \dfrac{\overline{\;\bar{B}\ ,\ B\;}}{\bar{B}\oplus\bar{A}\ ,\ B}$$



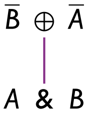
ALL: Coalescence

$$\dfrac{\dfrac{}{\overline{A}\,,\ A}}{\overline{B}\oplus\overline{A}\,,\ A} \qquad \dfrac{\dfrac{}{\overline{B}\,,\ B}}{\overline{B}\oplus\overline{A}\,,\ B}$$

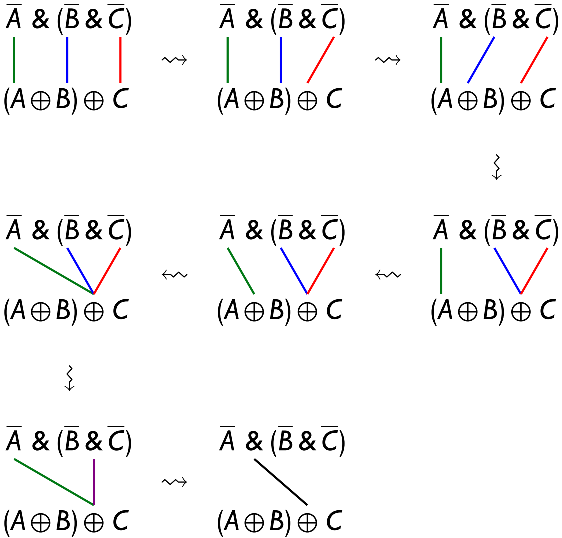


ALL: Coalescence

$$\dfrac{\dfrac{\overline{\overline{A}\,,\;A}}{\overline{B}\oplus\overline{A}\,,\;A} \qquad \dfrac{\overline{\overline{B}\,,\;B}}{\overline{B}\oplus\overline{A}\,,\;B}}{\overline{B}\oplus\overline{A}\,,\;A\,\&\,B}$$



ALL: Coalescence



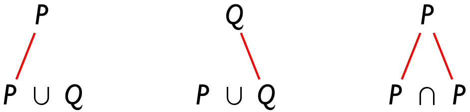
Remark

The set of subsets of a set X ordered by inclusion ($\subseteq$)

- Is a free distributive lattice: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Models ALL:

$$\begin{array}{ccc} A \vdash B & \Rightarrow & A \subseteq B \\ A \mathbin{\&} B & \Rightarrow & A \cap B \\ A \oplus B & \Rightarrow & A \cup B \end{array}$$

- Correctness: every resolution contains at least one link



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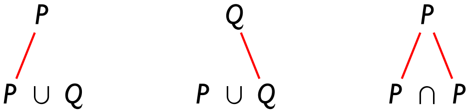
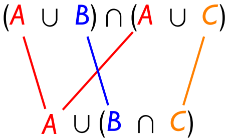


But distributivity destroys coalescence:



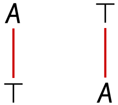
Properties of ALL proof nets

De-sequentialization	$P \Longrightarrow N$	✓ linear-time
Sequentialization / correctness	$N \Longrightarrow P$	✓ linear-time
Composition / cut-elimination	$N_1 \circ N_2$	✓ P-time
Proof equivalence / canonicity	$\begin{array}{ccc} P_1 & \stackrel{?}{\sim} & P_2 \\ \Downarrow & & \Downarrow \\ N_1 & = & N_2 \end{array}$	✓

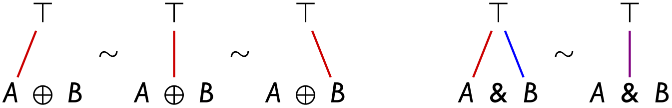
ALLU

Additive units: 0, $\top$

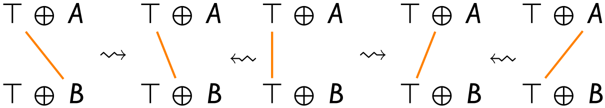
$$\overline{A, \top}$$



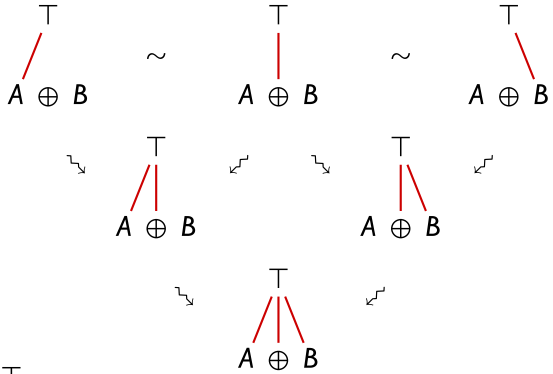
Equivalence:

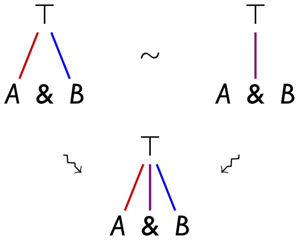


Non-confluence:



Saturation





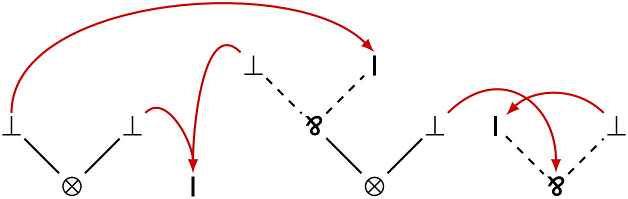
Properties of ALLU proof nets

De-sequentialization	$P \Longrightarrow N$	✓ linear-time
Sequentialization / correctness	$N \Longrightarrow P$	✓ linear-time
Composition / cut-elimination	$N_1 \circ N_2$	✓ P-time
Proof equivalence / canonicity	$\begin{array}{ccc} P_1 & \overset{?}{\sim} & P_2 \\ \Downarrow & & \Downarrow \\ N_1 & = & N_2 \end{array}$	✓

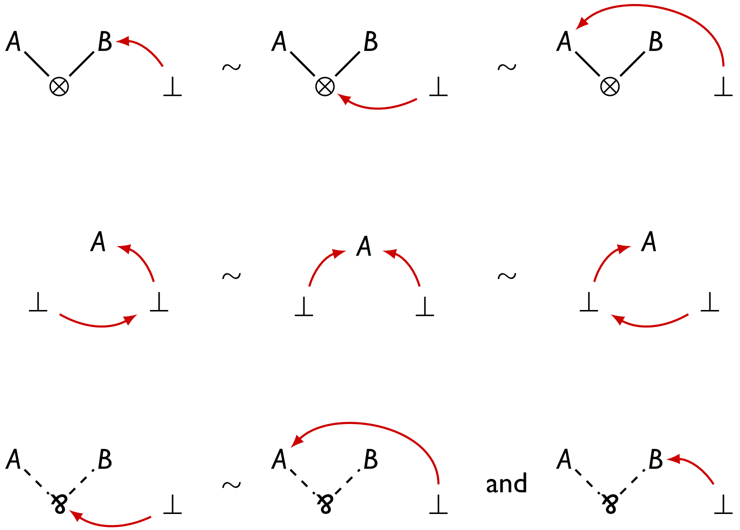
MLLU

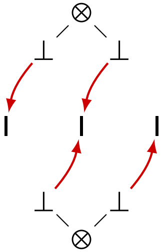
$$A, B, C \ ::= \ \mathbf{I} \ \mid \ \bot \ \mid \ A \otimes B \ \mid \ A \parr B$$

$$\frac{}{\mathbf{I}} \qquad \frac{\Gamma}{\Gamma, \bot} \qquad \frac{\Gamma, A \qquad B, \Delta}{\Gamma, A \otimes B, \Delta} \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}$$

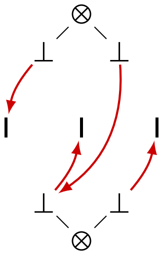


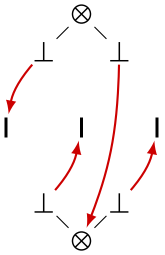
Equivalence

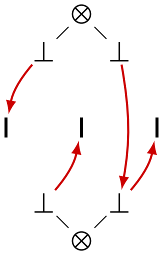


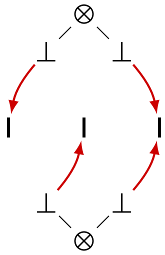


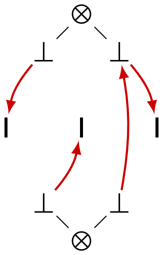
$$\bot \otimes \bot, \mathbf{I}, \mathbf{I}, \mathbf{I}, \bot \otimes \bot$$

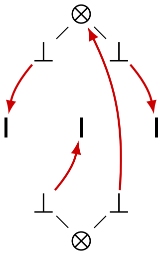


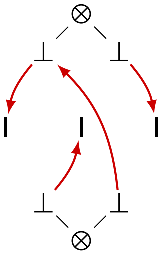


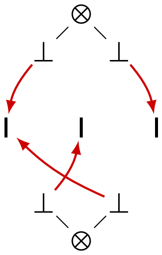


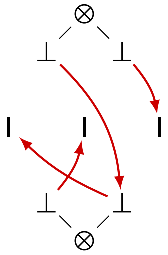


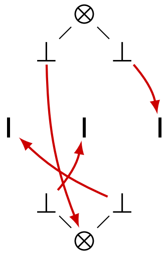


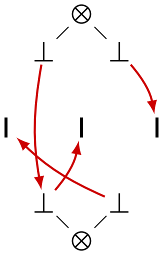


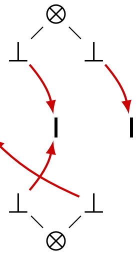












Properties of MLLU proof nets

De-sequentialization	$P \Longrightarrow N$	✓ linear-time
Sequentialization / correctness	$N \Longrightarrow P$	✓ linear-time
Composition / cut-elimination	$N_1 \circ N_2$	✓ P-time
Proof equivalence / canonicity	$\begin{array}{ccc} P_1 & \stackrel{?}{\sim} & P_2 \\ \Downarrow & & \Downarrow \\ N_1 & = & N_2 \end{array}$	✖ PSPACE

Proof nets and complexity

	MLL	MLLU	ALL	ALLU
De-sequentialization	✓	✓	✓	✓
Sequentialization / correctness	✓[2]	✓	✓[6]	✓[6]
Composition / cut-elimination	✓	✓	✓	✓
Proof equivalence / canonicity	✓[1]	✖[5]	✓[3]	✓[4]

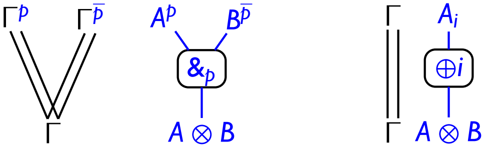
[1][Girard 1987]; [2][Guerrini 1999]; [3][Hu 1999];

[4][H 2011]; [5][H & Houston 2014]; [6][H & Hughes 2015]

MALL

Monomial nets

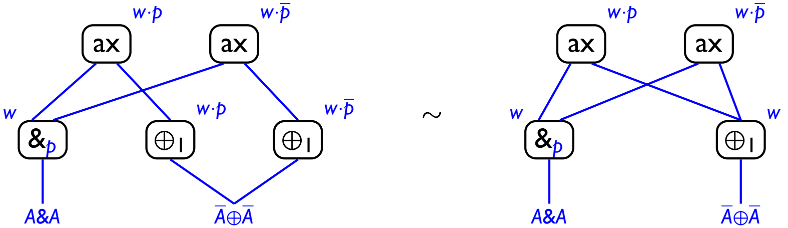
$$\frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \,\&\, B} \qquad\qquad \frac{\Gamma, A_i}{\Gamma, A_1 \oplus A_2}$$



Links are indexed by monomial weights: elements

$$p_1 \cdot p_2 \cdot p_3 \ldots p_n \;\cdot\; \overline{q}_1 \cdot \overline{q}_2 \cdot \overline{q}_3 \ldots \overline{q}_m$$

from a boolean algebra $(P, 0, 1, +, \cdot, \overline{})$ whose atomic elements $p, \overline{p} \in P$ indicate the two branches of a subformula $A \,\&_p\, B$



Distributivity: $(w \cdot p) + (w \cdot \overline{p}) \;=\; w \cdot (p + \overline{p}) \;=\; w \cdot 1 \;=\; w$

MALL proof nets

M: Monomial nets [Girard 1996, Laurent & Maieli 2008]

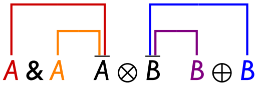
S: Slice nets [Hughes & Van Glabbeek 2005]

C: Conflict nets [Hughes & H 2016]

	M	**S**	**C**
De-sequentialization	✓		
Sequentialization / correctness	✖		
Composition / cut-elimination	?		
Proof equivalence / canonicity	✖		

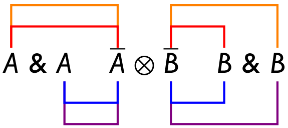
Sequent + linking

$$\dfrac{\dfrac{\overline{A\,,\,\bar{A}} \qquad \overline{\bar{B}\,,\,B}}{A\,,\,\bar{A}\otimes\bar{B}\,,\,B\oplus B} \qquad \dfrac{\overline{A\,,\,\bar{A}} \qquad \overline{\bar{B}\,,\,B}}{A\,,\,\bar{A}\otimes\bar{B}\,,\,B\oplus B}}{A\,\&\,A\,,\,\bar{A}\otimes\bar{B}\,,\,B\oplus B}$$



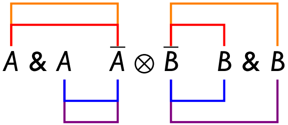
$$\dfrac{\dfrac{\overline{A\,,\,\bar{A}} \qquad \overline{\bar{B}\,,\,B}}{A\,,\,\bar{A}\otimes\bar{B}\,,\,B\oplus B} \qquad \dfrac{\overline{A\,,\,\bar{A}} \qquad \overline{\bar{B}\,,\,B}}{A\,,\,\bar{A}\otimes\bar{B}\,,\,B\oplus B}}{A\,\&\,A\,,\,\bar{A}\otimes\bar{B}\,,\,B\oplus B}$$

Slice nets

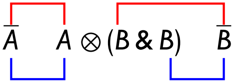


A set of links for each slice

Slice nets



A set of links for each slice



$$\frac{\overline{\bar{A}, A} \qquad \dfrac{\overline{B, \bar{B}} \qquad \overline{B, \bar{B}}}{B \,\&\, B, \bar{B}}}{\bar{A}, A \otimes (B \,\&\, B), \bar{B}}$$

But there may be 2^n slices, for n the number of &-occurrences

MALL proof nets

M:	Monomial nets	[Girard 1996, Laurent & Maieli 2008]
S:	Slice nets	[Hughes & Van Glabbeek 2005]
C:	Conflict nets	[Hughes & H 2016]

	M	**S**	**C**
De-sequentialization	✓	✖	
Sequentialization / correctness	✖	✓	
Composition / cut-elimination	?	✓	
Proof equivalence / canonicity	✖	✓	

The problem: size v canonicity

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\,,C\end{matrix} \qquad \begin{matrix}\Pi_3\\ D\end{matrix}}{A\,,\,C \otimes D} \qquad \dfrac{\begin{matrix}\Pi_2\\ B\,,C\end{matrix} \qquad \begin{matrix}\Pi_3\\ D\end{matrix}}{B\,,\,C \otimes D}}{A \mathbin{\&} B\,,\,C \otimes D} \quad \sim \quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\,,C\end{matrix} \qquad \begin{matrix}\Pi_2\\ B\,,C\end{matrix}}{A \mathbin{\&} B\,,\,C} \qquad \begin{matrix}\Pi_3\\ D\end{matrix}}{A \mathbin{\&} B\,,\,C \otimes D}$$

Distributivity:

$$\underline{(a \cdot x) + (a \cdot b)} \qquad (a \cdot x) + \overline{(a \cdot b) + (y \cdot b)}$$

$$\swarrow \qquad\qquad \searrow$$

$$\underline{(a \cdot (x + b))} + (y \cdot b) \qquad\qquad (a \cdot x) + \overline{((a + y) \cdot b)}$$

$$\dfrac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \parr D}\parr}{\Gamma, A \parr B, C \parr D}\parr \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A \parr B, C, D}\parr}{\Gamma, A \parr B, C \parr D}\parr$$

$$\dfrac{\dfrac{\Gamma, A, C \quad \Gamma, A, D}{\Gamma, A, C \,\&\, D}\& \qquad \dfrac{\Gamma, B, C \quad \Gamma, B, D}{\Gamma, B, C \,\&\, D}\&}{\Gamma, A \,\&\, B, C \,\&\, D}\& \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A, C \quad \Gamma, B, C}{\Gamma, A \,\&\, B, C}\& \qquad \dfrac{\Gamma, A, D \quad \Gamma, B, D}{\Gamma, A \,\&\, B, D}\&}{\Gamma, A \,\&\, B, C \,\&\, D}\&$$

$$\dfrac{\Gamma, A \qquad \dfrac{B, \Delta, C \quad D, \Sigma}{B, \Delta, C \otimes D, \Sigma}\otimes}{\Gamma, A \otimes B, \Delta, C \otimes D, \Sigma}\otimes \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A \quad B, \Delta, C}{\Gamma, A \otimes B, \Delta, C}\otimes \qquad D, \Sigma}{\Gamma, A \otimes B, \Delta, C \otimes D, \Sigma}\otimes$$

$$\dfrac{\dfrac{\Gamma, A_i, B_j}{\Gamma, A_1, B_1 \oplus B_2}\oplus_j}{\Gamma, A_1 \oplus A_2, B_1 \oplus B_2}\oplus_i \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A_i, B_j}{\Gamma, A_1 \oplus A_2, B, C_j}\oplus_i}{\Gamma, A_1 \oplus A_2, B_1 \oplus B_2}\oplus_j$$

$$\dfrac{\Gamma, A \qquad \dfrac{B, \Delta, C, D}{B, \Delta, C \parr D}\parr}{\Gamma, A \otimes B, \Delta, C \parr D}\otimes \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A \quad B, \Delta, C, D}{\Gamma, A \otimes B, \Delta, C, D}\otimes}{\Gamma, A \otimes B, \Delta, C \parr D}\parr$$

$$\dfrac{\dfrac{\Gamma, A, C_i \quad \Gamma, B, C_i}{\Gamma, A \,\&\, B, C_i}\&}{\Gamma, A \,\&\, B, C_1 \oplus C_2}\oplus_i \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A, C_i}{\Gamma, A, C_1 \oplus C_2}\oplus_i \qquad \dfrac{\Gamma, B, C_i}{\Gamma, B, C_1 \oplus C_2}\oplus_i}{\Gamma, A \,\&\, B, C_1 \oplus C_2}\&$$

$$\dfrac{\Gamma, A \qquad \dfrac{B, \Delta, C_i}{B, \Delta, C_1 \oplus C_2}\oplus_i}{\Gamma, A \otimes B, \Delta, C_1 \oplus C_2}\otimes \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A \quad B, \Delta, C_i}{\Gamma, A \otimes B, \Delta, C_i}\otimes}{\Gamma, A \otimes B, \Delta, C_1 \oplus C_2}\oplus_i$$

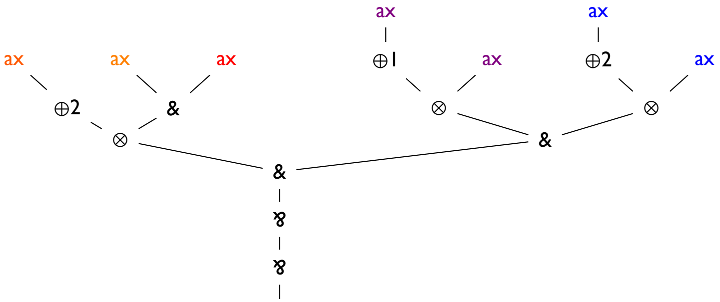
$$\dfrac{\dfrac{\Gamma, A_i, B, C}{\Gamma, A_1, B \parr C}\parr}{\Gamma, A_1 \oplus A_2, B \parr C}\oplus_i \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A_i, B, C}{\Gamma, A_1 \oplus A_2, B, C}\oplus_i}{\Gamma, A_1 \oplus A_2, B \parr C}\parr$$

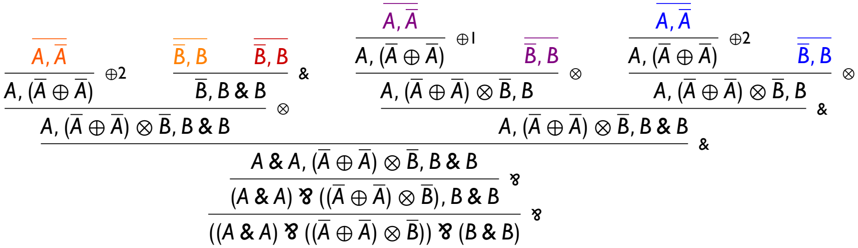
$$\dfrac{\dfrac{\Gamma, A, B, C \quad \Gamma, A, B, D}{\Gamma, A, B, C \,\&\, D}\&}{\Gamma, A \parr B, C \,\&\, D}\parr \quad\leftrightarrow\quad \dfrac{\dfrac{\Gamma, A, B, C}{\Gamma, A \parr B, C}\parr \qquad \dfrac{\Gamma, A, B, D}{\Gamma, A \parr B, D}\parr}{\Gamma, A \parr B, C \,\&\, D}\&$$

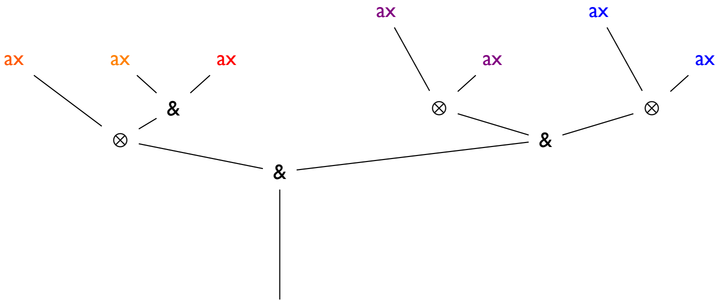
Conflict nets: idea

(strong) canonicity:	invariance under all commutations
local canonicity:	invariance under local commutations

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\ \oplus 2
      \qquad
      \dfrac{\dfrac{}{\overline{B}, B} \quad \dfrac{}{\overline{B}, B}}{\overline{B}, B \,\&\, B}\ \&
    }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}\ \otimes
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\ \oplus 1
        \quad
        \dfrac{}{\overline{B}, B}
      }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\ \otimes
      \qquad
      \dfrac{
        \dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\ \oplus 2
        \quad
        \dfrac{}{\overline{B}, B}
      }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\ \otimes
    }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}\ \&
  }{
    \dfrac{A \,\&\, A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}{(A \,\&\, A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B}), B \,\&\, B}\ \parr
  }\ \&
}{((A \,\&\, A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B})) \parr (B \,\&\, B)}\ \parr
$$




$A, \overline{A}$
$A, (\overline{A} \oplus \overline{A})$ ⊕2
$\overline{B}, B$
$\overline{B}, B$
$\overline{B}, B \& B$ &
$A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \& B$ ⊗
$A, \overline{A}$
$A, (\overline{A} \oplus \overline{A})$ ⊕1
$\overline{B}, B$
$A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B$ ⊗
$A, \overline{A}$
$A, (\overline{A} \oplus \overline{A})$ ⊕2
$\overline{B}, B$
$A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B$ ⊗
$A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \& B$ &
$A \& A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \& B$ &
$(A \& A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B}), B \& B$ ⅋
$((A \& A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B})) \parr (B \& B)$ ⅋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\dfrac{\dfrac{\dfrac{\dfrac{\overline{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\oplus 2 \qquad \dfrac{\overline{\overline{B}, B} \qquad \overline{\overline{B}, B}}{\overline{B}, B \,\&\, B}\,\&}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}\otimes \qquad \dfrac{\dfrac{\dfrac{\overline{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\oplus 1 \qquad \overline{\overline{B}, B}}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\otimes \qquad \dfrac{\dfrac{\overline{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\oplus 2 \qquad \overline{\overline{B}, B}}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\otimes}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}\,\&}{\dfrac{A \,\&\, A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}{(A \,\&\, A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B}), B \,\&\, B}\parr}\,\&}{((A \,\&\, A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B})) \parr (B \,\&\, B)}\parr
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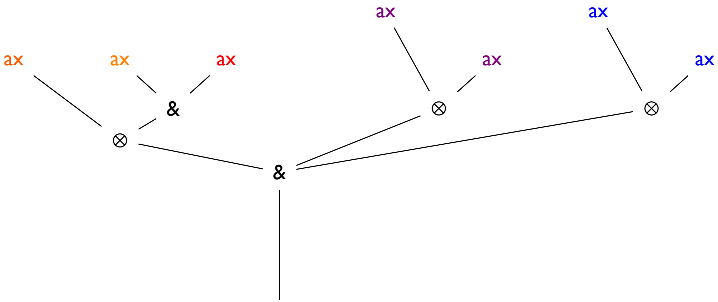

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      \dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\oplus 2
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      \dfrac{\dfrac{}{\overline{B}, B} \qquad \dfrac{}{\overline{B}, B}}{\overline{B}, B \,\&\, B}\,\&
    }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}\otimes
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      \dfrac{
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        \qquad
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      }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\otimes
      \qquad
      \dfrac{
        \dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\oplus 2
        \qquad
        \dfrac{}{\overline{B}, B}
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    }{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}\,\&
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    \dfrac{A \,\&\, A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \,\&\, B}{(A \,\&\, A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B}), B \,\&\, B}\parr
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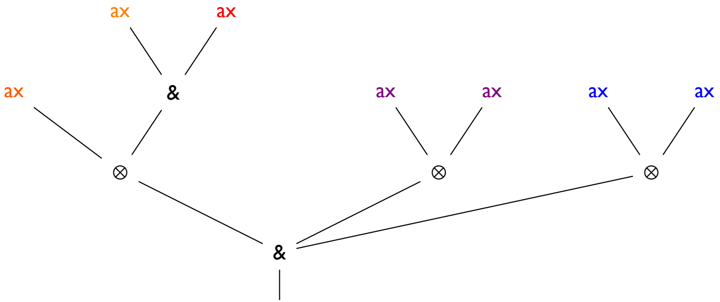

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}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \mathbin{\&} B}\;\otimes
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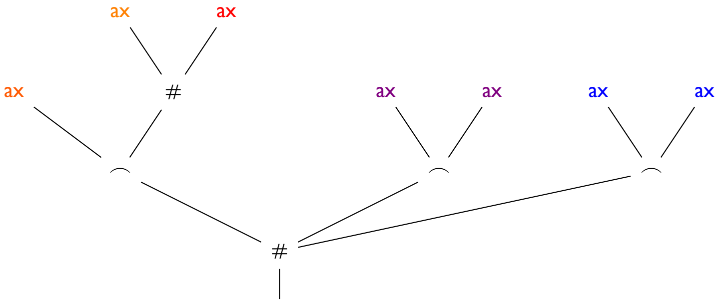
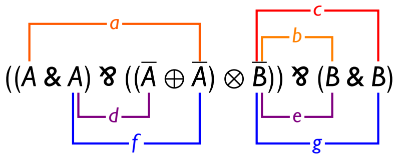
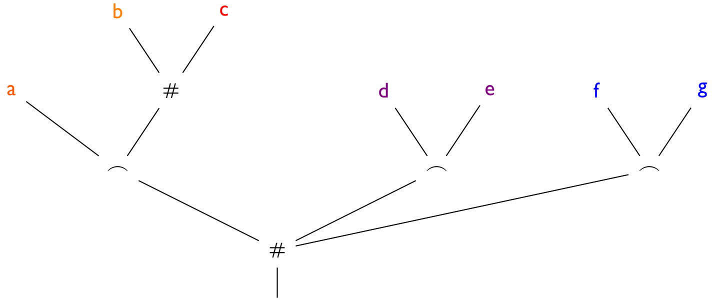
\dfrac{\dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\;\oplus 1 \quad \dfrac{}{\overline{B}, B}}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\;\otimes
\quad
\dfrac{\dfrac{\dfrac{}{A, \overline{A}}}{A, (\overline{A} \oplus \overline{A})}\;\oplus 2 \quad \dfrac{}{\overline{B}, B}}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B}\;\otimes
}{A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \mathbin{\&} B}\;\&
}{A \mathbin{\&} A, (\overline{A} \oplus \overline{A}) \otimes \overline{B}, B \mathbin{\&} B}\;\&
}{(A \mathbin{\&} A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B}), B \mathbin{\&} B}\;\parr
}{((A \mathbin{\&} A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B})) \parr (B \mathbin{\&} B)}\;\parr
$$





a
c
b
$((A \mathbin{\&} A) \parr ((\overline{A} \oplus \overline{A}) \otimes \overline{B})) \parr (B \mathbin{\&} B)$
d
e
f
g



b
c
a
#
d
e
f
g
#

Conflict nets

Data: ($\#/\frown$) alternating, n-ary conflict tree T

$$T ::= \Delta \subseteq \Gamma \mid (T \# \cdots \# T) \mid (T \frown \cdots \frown T)$$

over an axiom linking ($\Delta = a, \overline{a}$) over a sequent Γ

Hybrid of focussing and proof nets:

- a conflict node $\#$ represents an ALL + $\parr$ proof net ($\&, \oplus, \parr$)
- a concord node $\frown$ represents an MLL + $\oplus$ proof net ($\otimes, \oplus, \parr$)
- ($\oplus/\parr$) are not confined to a layer

Correctness / sequentialization: by coalescence

De-sequentialization

$$\left[\!\left[\overline{a, \overline{a}}\right]\!\right] = (a, \overline{a})$$

$$\left[\!\!\left[\dfrac{\begin{matrix}\Pi \\ \Gamma, A, B\end{matrix}}{\Gamma, A \parr B}\right]\!\!\right] = [\![\Pi]\!]$$

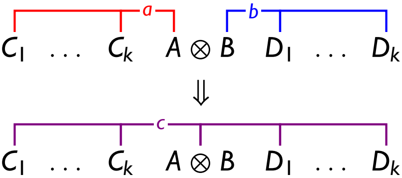
$$\left[\!\!\left[\dfrac{\begin{matrix}\Pi \\ \Gamma, A\end{matrix}}{\Gamma, A \oplus B}\right]\!\!\right] = [\![\Pi]\!]$$

$$\left[\!\!\left[\dfrac{\begin{matrix}\Pi_1 \\ \Gamma, A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ B, \Delta\end{matrix}}{\Gamma, A \otimes B, \Delta}\right]\!\!\right] = [\![\Pi_1]\!] \frown [\![\Pi_2]\!]$$

$$\left[\!\!\left[\dfrac{\begin{matrix}\Pi_1 \\ \Gamma, A\end{matrix} \qquad \begin{matrix}\Pi_2 \\ \Gamma, B\end{matrix}}{\Gamma, A \,\&\, B}\right]\!\!\right] = [\![\Pi_1]\!] \,\#\, [\![\Pi_2]\!]$$

Coalescence: MLL + ⊕

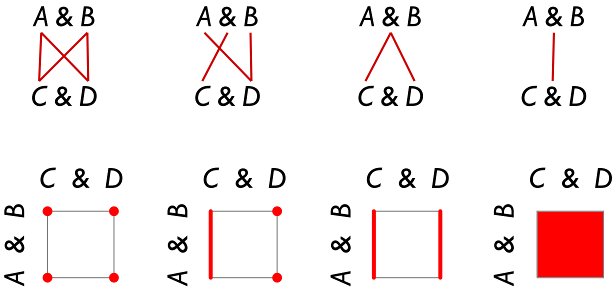
$$\frown(\Delta) \;\Rightarrow\; \Delta$$



$(a \frown b \frown T_1 \frown \cdots \frown T_n)$

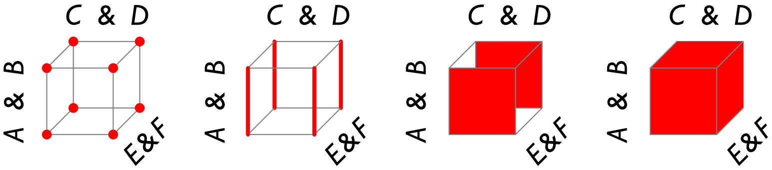
$(c \frown T_1 \frown \cdots \frown T_n)$

2D ALL coalescence



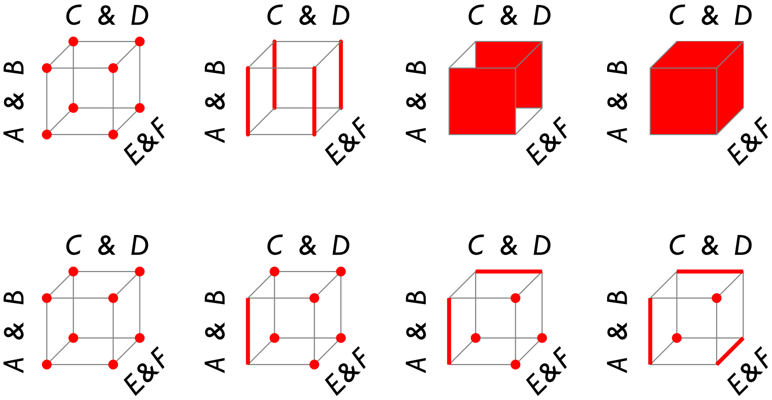
3D ALL coalescence

A & B , C & D , E & F



3D ALL coalescence

A & B , C & D , E & F



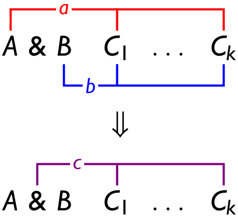
$$A \& B \;,\; C \& D \;,\; E \& F$$

$$\underline{(A, C, E) \# (B, C, E)} \;\#\; (A, C, F) \;\#\; \underline{(B, C, F) \# (B, D, F)} \;\#\; (B, D, E) \;\#\; \underline{(A, D, E) \# (A, D, F)}$$

$$\Downarrow$$

$$\underline{(A \& B, C, E)} \;\#\; (A, C, F) \;\#\; \underline{(B, C \& D, F)} \;\#\; (B, D, E) \;\#\; \underline{(A, D, E \& F)}$$

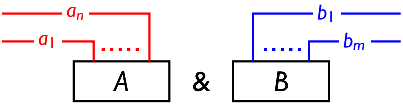
Coalescence: ALL + ⅋



$(a \# b)$

$\# (\Delta) \Rightarrow \Delta$

c

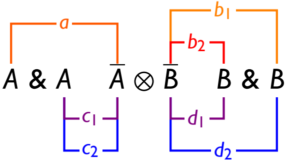


$$(a_1 \# \cdots \# a_n \# b_1 \# \ldots b_m)$$

$$\Downarrow$$

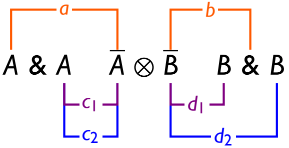
$$\big((a_1 \# \cdots \# a_n) \# (b_1 \# \ldots b_m)\big)$$

Example



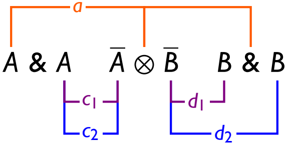
$$(a \frown (b_1 \# b_2)) \# (c_1 \frown d_1) \# (c_2 \frown d_2)$$

Example



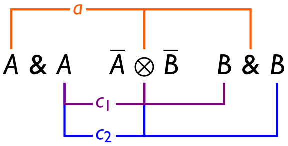
$$(a \frown b) \# (c_1 \frown d_1) \# (c_2 \frown d_2)$$

Example



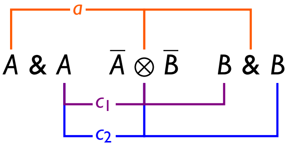
$$a \# (c_1 \frown d_1) \# (c_2 \frown d_2)$$

Example



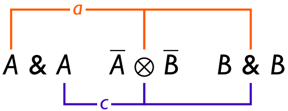
$$a \# c_1 \# c_2$$

Example



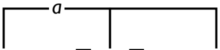
$$a \# (c_1 \# c_2)$$

Example



$$a \# c$$

Example



$A \mathbin{\&} A \qquad \overline{A} \otimes \overline{B} \qquad B \mathbin{\&} B$

a

MALL proof nets

M: Monomial nets [Girard 1996, Laurent & Maieli 2008]

S: Slice nets [Hughes & Van Glabbeek 2005]

C: Conflict nets [Hughes & H 2016]

	M	**S**	**C**
De-sequentialization	✓	✖	✓
Sequentialization / correctness	✖	✓	✓
Composition / cut-elimination	?	✓	✖
Proof equivalence / canonicity	✖	✓	~

Cut-elimination

Four rewrite steps:

- axiom–cut–axiom
- tensor–cut–par
- with–cut–plus
- conflict–cut or cut–conflict

Three logical steps and one duplication step

$$\dfrac{\dfrac{\overset{\Pi_1}{A\,,\,C} \qquad \overset{\Pi_2}{B\,,\,C}}{A \,\&\, B\,,\,C} \qquad \overset{\Pi_3}{\overline{C}\,,\,\Gamma}}{A \,\&\, B\,,\,\Gamma} \quad \sim \quad \dfrac{\dfrac{\overset{\Pi_1}{A\,,\,C} \qquad \overset{\Pi_3}{\overline{C}\,,\,\Gamma}}{A\,,\,\Gamma} \qquad \dfrac{\overset{\Pi_2}{B\,,\,C} \qquad \overset{\Pi_3}{\overline{C}\,,\,\Gamma}}{B\,,\,\Gamma}}{A \,\&\, B\,,\,\Gamma}$$

Bureaucracy or computation?

Thank you