Semidiscrete Toda lattices

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March 3, 2012
Introduction

Lax pairs in the continuous case

Symmetry approach in the continuous case

Lax pairs in the semidiscrete case

Symmetry approach in the semidiscrete case
The Toda chain

\[ q_{xx}(j) = \exp(q(j + 1) - q(j)) - \exp(q(j) - q(j - 1)). \]
The Toda chain

- \( q_{xx}(j) = \exp(q(j + 1) - q(j)) - \exp(q(j) - q(j - 1)). \)
- Change of variables:
  \[
  (\ln h(j))_{xx} = h(j + 1) - 2h(j) + h(j - 1),
  \]
  where \( h(j) = \exp(q(j + 1) - q(j)). \)
- Another change of variables:
  \[
  u_{xx}(j) = \exp(u(j + 1) - 2u(j) + u(j - 1)),
  \]
  where \( h(j) = u_{xx}(j). \)
How to obtain a finite dynamical system?

- Trivial boundary conditions: $u(-1) = u(r) = -\infty$.
- Periodic boundary conditions: $u(j + r) = u(j)$. 

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Semidiscrete Toda lattices
How to obtain a finite dynamical system?

- Trivial boundary conditions: \( u(-1) = u(r) = -\infty \).
- Periodic boundary conditions: \( u(j + r) = u(j) \).
- Any other ideas?
Integrable boundary conditions

Matrix form: \( u_{xx} = \exp(Ku) \), where

\[
K = \begin{pmatrix}
-2 & 1 & 0 & \ldots & 0 \\
1 & -2 & 1 & \ldots & 0 \\
0 & 1 & -2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -2
\end{pmatrix}
\]
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- Notice that $-K$ is the Cartan matrix of an $A$-series Lie algebra.
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\end{pmatrix}.
\]

- Notice that \(-K\) is the Cartan matrix of an \(A\)-series Lie algebra.

- There are other simple Lie algebras!

- Bogoyavlensky (1976): \( u_{xx} = \exp(Ku) \) is integrable if \(-K\) is the Cartan matrix of a simple Lie algebra.
Two-dimensional Toda lattice

- $q_{xy}(j) = \exp(q(j + 1) - q(j)) - \exp(q(j) - q(j - 1))$
- or
  \[(\ln h(j))_{xy} = h(j + 1) - 2h(j) + h(j - 1),\]
  where $h(j) = \exp(q(j + 1) - q(j))$,
- or
  $u(j)_{xy} = \exp(u(j + 1) - 2u(j) + u(j - 1))$,
  where $h(j) = u_{xy}(j)$,
- or $u_{xy} = \exp(Ku)$ (exponential system).
Integrability in two-dimensional case

Various approaches by
- Leznov (1980)
- Shabat, Yamilov (1981)
- and probably by many others...

give the same result:

*Boundary conditions corresponding to simple Lie algebras are integrable whatever definition of integrability is being used.*
Laplace invariants

Definition

Functions $h = b_y - ab - c$ and $k = a_x - ab - c$ are called the Laplace invariants of the hyperbolic differential operator

$$\mathcal{L} = \partial_x \partial_y + a \partial_x + b \partial_y + c.$$
Laplace invariants

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\[
\mathcal{L} = \partial_x \partial_y + a \partial_x + b \partial_y + c.
\]

Lemma
Two hyperbolic differential operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are gauge-equivalent (i.e. \( \mathcal{L}_2 = \omega^{-1} \mathcal{L}_1 \omega \) for some function \( \omega = \omega(x, y) \)) iff their Laplace invariants are equal.
DLT and Toda lattice

Consider a sequence of hyperbolic operators

$$\mathcal{L}_j = \partial_x \partial_y + a(j)\partial_x + b(j)\partial_y + c(j)$$

such that any two neighboring operators are linked by a *Darboux-Laplace transformation*:

$$\mathcal{L}_{j+1} \mathcal{D}_j = \mathcal{D}_{j+1} \mathcal{L}_j,$$

where $\mathcal{D}_j = \partial_x + b(j)$. Then the Laplace invariants of these operators satisfy the two-dimensional Toda lattice:

$$k(j + 1) = h(j), \quad h(j + 1) = 2h(j) - k(j) + (\ln h(j))_{xy}.$$
Lax presentations for infinite lattice

Darboux-Laplace transformation

\[ D_j = \partial_x + b(j) : \mathcal{L}_j \to \mathcal{L}_{j+1} \]

leads to the following Lax presentation for the Toda lattice:

\[
\left\{ \begin{array}{l}
\psi_x(j) = \psi(j + 1) + q_x(j)\psi(j) \\
\psi_y(j) = -h(j - 1)\psi(j - 1)
\end{array} \right.,
\]

where \( h(j) = \exp(q(j + 1) - q(j)) \).
Lax presentations for infinite lattice

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\psi_x(j) &= \psi(j + 1) + q_x(j)\psi(j) \\
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\end{aligned}
\]

where \( h(j) = \exp(q(j + 1) - q(j)) \).

Symmetry \( x \leftrightarrow y \) provides another Lax presentation:

\[
\begin{aligned}
\varphi_x(j) &= -h(j - 1)\varphi(j - 1) \\
\varphi_y(j) &= \varphi(j + 1) + q_y(j)\varphi(j)
\end{aligned}
\]
Lax pair consistent with a boundary condition

Consider boundary condition of the type

\[ q(-1) = F(q(0), q(1), \ldots, q(k)) \]

In order to obtain a Lax presentation for Toda lattice satisfying this boundary condition, one has to express \( \psi(-1) \) in terms of \( \psi(0), \psi(1), \ldots \). But this works only for the trivial boundary condition \( q(-1) = \infty \).
Lax pair consistent with a boundary condition

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In order to obtain a Lax presentation for Toda lattice satisfying this boundary condition, one has to express \( \psi(-1) \) in terms of \( \psi(0), \psi(1), \ldots \). But this works only for the trivial boundary condition \( q(-1) = \infty \).

Habibullin’s idea (2005): why not to express \( \psi(-1) \) and \( \varphi(-1) \) in terms of both sets of variables:

\[ \psi(0), \psi(1), \ldots \quad \text{and} \quad \varphi(0), \varphi(1), \ldots ? \]
Lax pairs for $A - D$ series lattices

Theorem
The $A - D$ series Toda lattices admit Lax presentation of the form

$$\begin{pmatrix} \Psi \\ \Phi \end{pmatrix}_x = \begin{pmatrix} A & K \\ M & C \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}, \quad \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}_y = \begin{pmatrix} B & L \\ N & D \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix},$$
Lax pairs for $A - D$ series lattices

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\begin{pmatrix}
\Psi \\
\Phi
\end{pmatrix}_x = \begin{pmatrix}
A & K \\
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\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Phi
\end{pmatrix},
\begin{pmatrix}
\Psi \\
\Phi
\end{pmatrix}_y = \begin{pmatrix}
B & L \\
N & D
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Phi
\end{pmatrix},
\]

i.e. for any Toda lattice there exist integers $k < k_1$ and $m < m_1$ such that Toda equations together with corresponding boundary conditions are equivalent to the compatibility conditions for the above matrix equations, where

\[
\Psi = (\psi(k + 1), \psi(k + 2), \ldots, \psi(k_1 - 1))^t, \\
\Phi = (\varphi(m + 1), \varphi(m + 2), \ldots, \varphi(m_1 - 1))^t;
\]

and matrices $A, B, \ldots, N$ are as follows:
Lax pairs in the continuous case

Symmetry approach in the continuous case

Lax pairs in the semidiscrete case

Symmetry approach in the semidiscrete case

\[
A = \begin{pmatrix}
q_x(k + 1) & 1 & 0 & \ldots & 0 \\
0 & q_x(k + 2) & 1 & \ldots & 0 \\
\vdots & & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 1 \\
A_{k+1} & A_{k+2} & A_{k+3} & \ldots & A_{k_1-1}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
B_{k+1} & B_{k+2} & \ldots & 0 & B_{k_1-1} \\
-h(k + 1) & 0 & \ldots & 0 & 0 \\
\vdots & & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & -h(k_1 - 2) & 0
\end{pmatrix},
\]
**Outline**
- Introduction
- Lax pairs in the continuous case
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\[
C = \begin{pmatrix}
C_{m+1} & C_{m+2} & \ldots & 0 & C_{m_1-1} \\
-h(m+1) & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & -h(m_1-2) & 0 \\
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
q_y(m+1) & 1 & 0 & \ldots & 0 \\
0 & q_y(m+2) & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
D_{m+1} & D_{m+2} & D_{m+3} & \ldots & D_{m_1-1} \\
\end{pmatrix},
\]

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\[
K = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \\
K_{m+1} & K_{m+2} & \ldots & K_{m_1-1}
\end{pmatrix}, \quad L = \begin{pmatrix}
L_{m+1} & L_{m+2} & \ldots & L_{m_1-1} \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \\
0 & 0 & \ldots & 0
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
M_{k+1} & M_{k+2} & \ldots & M_{k_1-1} \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0
\end{pmatrix}, \quad N = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \\
0 & 0 & \ldots & 0 \\
N_{k+1} & N_{k+2} & \ldots & N_{k_1-1}
\end{pmatrix}.
\]
Remarks

- Since boundary conditions on the left and on the right are independent, this approach provides Lax presentations for a wider class of systems than just $A - D$ series Toda lattices.
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▶ Since boundary conditions on the left and on the right are independent, this approach provides Lax presentations for a wider class of systems than just $A - D$ series Toda lattices.

▶ Moreover, Lax presentations of the above form lead to systems that are not necessarily exponential.
Remarks

- Since boundary conditions on the left and on the right are independent, this approach provides Lax presentations for a wider class of systems than just $A-D$ series Toda lattices.
- Moreover, Lax presentations of the above form lead to systems that are not necessarily exponential.
- If one seeks for Lax presentation for some boundary value problem for the Toda lattice, this approach gives no hint on how to choose indeterminate matrix elements.
Boundary conditions and involutions

In terms of the variables $h(j)$ all $A - C$ series Toda lattices are obtained by the trivial boundary condition $h(r) = 0$ on the right edge and by the following boundary conditions on the left edge:

- $h(-1) = 0$ for $A$-series (trivial);
- $h(-j) = h(j - 1)$ for $B$-series (involution);
- $h(-j) = h(j)$ for $C$-series (involution).

Boundary condition for the $D$-series Toda lattice can not be expressed in terms of the variables $h(j)$. 
Motivation for Habibullin’s idea

Each of the wave functions $\psi(j)$ and $\varphi(j)$ satisfy the hyperbolic differential equation $L_j^\psi \psi(j) = 0$ ($L_j^\varphi \varphi(j) = 0$ resp.) In $B$- and $C$-series cases there are relations between the Laplace invariants of these equations.
\textbf{B-series:}

\[
\begin{align*}
    h^\varphi(0) &= h(-1) = h(2) = h^\psi(2), & k^\varphi(0) &= h(0) = h(1) = k^\psi(2), \\
    h^\psi(1) &= h(1) = h(0) = h^\varphi(1), & k^\psi(1) &= h(1) = h(0) = k^\varphi(1).
\end{align*}
\]
\[ h^{\varphi}(0) = h(-1) = h(2) = h^{\psi}(2), \quad k^{\varphi}(0) = h(0) = h(1) = k^{\psi}(2), \]
\[ h^{\psi}(1) = h(1) = h(0) = h^{\varphi}(1), \quad k^{\psi}(1) = h(1) = h(0) = k^{\varphi}(1). \]

This means that pairs of operators \( \mathcal{L}_0^{\varphi} \) and \( \mathcal{L}_2^{\psi}, \mathcal{L}_1^{\psi} \) and \( \mathcal{L}_1^{\varphi} \) are gauge-equivalent. Therefore, there exist the multipliers \( R = R(x, y) \) and \( S = S(x, y) \), such that
\[ \varphi(0) = R \psi(2), \quad \psi(1) = S \varphi(1). \]
C-series:

\[ h^{\varphi}(0) = h(-1) = h(1) = h^{\psi}(1), \quad k^{\varphi}(0) = h(0) = k^{\psi}(1), \]
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C-series:

\[ h^\varphi(0) = h(-1) = h(1) = h^\psi(1), \quad k^\varphi(0) = h(0) = k^\psi(1), \]
\[ k^\psi(0) = h(-1) = h(1) = k^\varphi(1), \quad h^\psi(0) = h(0) = h^\varphi(1). \]

Similarly, these relations imply the existence of \( R = R(x, y) \), and \( S = S(x, y) \) such that

\[ \varphi(0) = R\psi(1), \quad \psi(0) = S\varphi(1). \]

Both examples are particular cases of Habibullin’s theorem.
Nonlocal variables

In one-dimensional case symmetries of the Toda chain are expressed in terms of the dynamical variables, but in two-dimensional case symmetries are expressed in terms of non-local variables (Shabat, 1995).
Nonlocal variables

In one-dimensional case symmetries of the Toda chain are expressed in terms of the dynamical variables, but in two-dimensional case symmetries are expressed in terms of non-local variables (Shabat, 1995). Rewrite the infinite Toda lattice as follows:

\[ b_y(j) = h(j) - h(j - 1), \]

where \( b(j) = q_x(j) \) and define the variables \( b^{(1)}(j) \) by

\[ \partial_x b(j) = b^{(1)}(j) - b^{(1)}(j - 1), \quad \partial_y b^{(1)}(j) = \partial_x h(j). \]
Nonlocal variables

In one-dimensional case symmetries of the Toda chain are expressed in terms of the dynamical variables, but in two-dimensional case symmetries are expressed in terms of non-local variables (Shabat, 1995). Rewrite the infinite Toda lattice as follows:

\[ b_y(j) = h(j) - h(j - 1), \]

where \( b(j) = q_x(j) \) and define the variables \( b^{(1)}(j) \) by

\[ \partial_x b(j) = b^{(1)}(j) - b^{(1)}(j - 1), \quad \partial_y b_n^{(1)}(j) = \partial_x h(j). \]

Consistency of these equations is provided by the Toda equation.
One has to introduce further non-localities in order to define $\partial_x b^{(1)}(j)$:

$$\partial_x b^{(1)}(j) = b^{(1)}(b(j + 1) - b(j)) + b^{(2)}(j) - b^{(2)}(j - 1);$$
One has to introduce further non-localities in order to define $\partial_x b^{(1)}(j)$:

$$\partial_x b^{(1)}(j) = b^{(1)}(b(j + 1) - b(j)) + b^{(2)}(j) - b^{(2)}(j - 1);$$

consistency leads to the following relation:

$$\partial_y b^{(2)}(j) = h(j)b^{(1)}(j + 1) - h(j + 1)b^{(1)}(j),$$

etc...
Lemma

Non-local variables $b^{(k)}(j)$, where $k = 2, 3, \ldots$, satisfy the following relations:

$$
\begin{align*}
\partial_y b^{(k)}(j) &= h(j) b^{(k-1)}(j + 1) - h(j + k - 1) b^{(k-1)}(j) \\
\partial_x b^{(k)}(j) &= b^{(k)}(j) (b(j + k) - b(j)) + b^{(k+1)}(j) - b^{(k+1)}(j - 1)
\end{align*}
$$

Consistency of these equations for any $k = 2, 3, \ldots$ is provided by the first of them for $k + 1$. 
Symmetries of the infinite lattice

The infinite Toda lattice admits the following symmetries:

- second order symmetry

\[ q_t(j) = b^2(j) + b^{(1)}(j) + b^{(1)}(j - 1); \]
Symmetries of the infinite lattice

The infinite Toda lattice admits the following symmetries:

- **second order symmetry**

  \[ q_t(j) = b^2(j) + b^{(1)}(j) + b^{(1)}(j - 1); \]

- **third order symmetry**

  \[
  q_t(j) = b^3(j) + b^{(2)}(j) + b^{(2)}(j - 1) + b^{(2)}(j - 2) + \\
  + b^{(1)}(j) (2b(j) + b(j + 1)) + \\
  + b^{(1)}(j - 1) (2b(j) + b(j - 1)) ;
  \]
Symmetries of the infinite lattice

The infinite Toda lattice admits the following symmetries:

- **second order symmetry**
  \[ q_t(j) = b^2(j) + b^{(1)}(j) + b^{(1)}(j - 1); \]

- **third order symmetry**
  \[
  q_t(j) = b^3(j) + b^{(2)}(j) + b^{(2)}(j - 1) + b^{(2)}(j - 2) + \\
  + b^{(1)}(j) (2b(j) + b(j + 1)) + \\
  + b^{(1)}(j - 1) (2b(j) + b(j - 1));
  \]

- **higher-order symmetries** are expressed in terms of higher non-localities.

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Semidiscrete Toda lattices
New set of variables

In the new set of variables

\[ u = e^{-q(-1)}, \quad v = e^{q(0)} \]

these symmetries are expressed in a more simple way:

\[ u_t = -u_{xx} - 2ru, \quad v_t = v_{xx} + 2rv, \]
\[ u_t = u_{xxx} + 3ru_x - 3su + 3r_xu, \quad v_t = v_{xxx} + 3rv_x + 3sv, \]

where \( r = b^{(1)}(0), \ s = b^{(2)}(0) + r(\ln v)_x \).
Classification theorem

What boundary conditions are compatible with the above symmetries?
Classification theorem

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- Only trivial boundary condition $u = 0$ is compatible with the second order symmetry.
Classification theorem

What boundary conditions are compatible with the above symmetries?

- Only trivial boundary condition \( u = 0 \) is compatible with the second order symmetry.

- Habibullin, Gürel (1997): if boundary condition \( u = F(v, v_x, v_y, v_{xy}) \) is compatible with the third order symmetry, then the function \( F \) belongs to one of the following classes:
  
  i) \( F(v, v_x, v_y, v_{xy}) = 0 \) (\( A \) series);
  
  ii) \( F(v, v_x, v_y, v_{xy}) = a \neq 0, a = const \) (\( B \) series);
  
  iii) \( F(v, v_x, v_y, v_{xy}) = av, a = const \neq 0 \) (\( C \) series);
  
  iv) \( F(v, v_x, v_y, v_{xy}) = \frac{v_{xy}}{a-v^2} + \frac{vv_xv_y}{(a-v^2)^2} \) (\( D \) series).
Classification theorem

What boundary conditions are compatible with the above symmetries?

- Only trivial boundary condition $u = 0$ is compatible with the second order symmetry.

- Habibullin, Gürel (1997): if boundary condition $u = F(v, v_x, v_y, v_{xy})$ is compatible with the third order symmetry, then the function $F$ belongs to one of the following classes:
  
  i) $F(v, v_x, v_y, v_{xy}) = 0$ (A series);
  
  ii) $F(v, v_x, v_y, v_{xy}) = a \neq 0, a = \text{const}$ (B series);
  
  iii) $F(v, v_x, v_y, v_{xy}) = av, a = \text{const} \neq 0$ (C series);
  
  iv) $F(v, v_x, v_y, v_{xy}) = \frac{v_{xy}}{a-v^2} + \frac{v v_x v_y}{(a-v^2)^2}$ (D series).

- More complicated boundary conditions lead to the systems that are not exponential.
Discrete analogs

There are several possible ways to discretize Toda systems.

▶ One-dimensional discrete Toda chains were considered by Suris (1990). Lax presentations were obtained for the chains corresponding to non-exceptional Lie algebras.
Discrete analogs

There are several possible ways to discretize Toda systems.

- One-dimensional discrete Toda chains were considered by Suris (1990). Lax presentations were obtained for the chains corresponding to non-exceptional Lie algebras.

- Purely discrete two-dimensional Toda lattices were considered by Ward (1995) and Habibullin (2005). Lax presentations for A- and C-series Toda lattices were obtained.
Discrete analogs

There are several possible ways to discretize Toda systems.

- One-dimensional discrete Toda chains were considered by Suris (1990). Lax presentations were obtained for the chains corresponding to non-exceptional Lie algebras.

- Purely discrete two-dimensional Toda lattices were considered by Ward (1995) and Habibullin (2005). Lax presentations for $A$- and $C$-series Toda lattices were obtained.

- Semidiscrete case mysteriously had not been considered before!
The purpose of the talk

The rest of this talk will concern the semidiscrete case, i.e. the Toda lattice with one of the independent variables being continuous and the other being discrete. The purpose of this talk is to highlight the similarities and the differences between the continuous case and the semidiscrete case.
Laplace invariants

Definition
Functions \( k_n(j) = \frac{c_n(j)}{a_n(j)} - (\ln a_n(j))' - b_n(j) \) and
\( h_n(j) = \frac{c_n(j)}{a_n(j)} - b_n(j - 1) \) are called the Laplace invariants of hyperbolic differential-difference operator

\[ \mathcal{L} = \partial_x T + a \partial_x + bT + c, \]

where \( T \) is the shift operator: \( T \psi_n = \psi_{n+1} \).
Laplace invariants

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$$\mathcal{L} = \partial_x T + a \partial_x + b T + c,$$

where $T$ is the shift operator: $T \psi_n = \psi_{n+1}$.

Lemma
Two hyperbolic differential-difference operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are gauge-equivalent (i.e. $\mathcal{L}_2 = (T \omega)^{-1} \mathcal{L}_1 \omega$ for some function $\omega = \omega_n(x)$) iff their Laplace invariants are equal.
DTL and Toda lattice

Consider the sequence of hyperbolic operators

\[ \mathcal{L}_j = \partial_x T + a_n(j) \partial_x + b_n(j) T + c_n(j) \]

such that any two neighboring operators are linked by *Darboux-Laplace transformation:*

\[ \mathcal{L}_{j+1} D_j = D_{j+1} \mathcal{L}_j, \]

where \( D_j = \partial_x + b_n(j). \)
Then the Laplace invariants of these operators satisfy the following equations:

\[
\begin{align*}
\left\{
\begin{array}{ll}
     k_n(j + 1) & = h_n(j) \\
     \left( \ln \frac{h_n(j)}{h_{n+1}(j)} \right)' & = h_{n+1}(j + 1) - h_{n+1}(j) - h_n(j) + h_n(j - 1) \\
\end{array}
\right.
\end{align*}
\]
Then the Laplace invariants of these operators satisfy the following equations:

\[
\begin{align*}
\left\{ 
  k_n(j + 1) &= h_n(j) \\
  \left( \ln \frac{h_n(j)}{h_{n+1}(j)} \right)' &= h_{n+1}(j + 1) - h_{n+1}(j) - h_n(j) + h_n(j - 1)
\end{align*}
\]

Introduce another set of variables by

\[
  h_n(j) = \exp(q_{n+1}(j + 1) - q_n(j));
\]

then the latter equation can be rewritten as follows:

\[
q_{n,x}(j) - q_{n+1,x}(j) = \exp(q_{n+1}(j + 1) - q_n(j)) - \exp(q_{n+1}(j) - q_n(j - 1)).
\]
Then the Laplace invariants of these operators satisfy the following equations:

\[
\begin{aligned}
&k_n(j + 1) = h_n(j) \\
&\left( \ln \frac{h_n(j)}{h_{n+1}(j)} \right)' = h_{n+1}(j + 1) - h_{n+1}(j) - h_n(j) + h_n(j - 1)
\end{aligned}
\]

Introduce another set of variables by

\[
h_n(j) = \exp(q_{n+1}(j + 1) - q_n(j));
\]

then the latter equation can be rewritten as follows:

\[
q_{n,x}(j) - q_{n+1,x}(j) = \exp(q_{n+1}(j+1) - q_n(j)) - \exp(q_{n+1}(j) - q_n(j-1)).
\]

These equations are two forms of the *semidiscrete Toda lattice*. 
Lax presentations for infinite lattice

Darboux-Laplace transformation

\[ \mathcal{D}_j = \partial_x + b(j) : \mathcal{L}_j \to \mathcal{L}_{j+1} \]

leads to the following Lax presentation for the Toda lattice:

\[
\begin{cases}
\psi_{n,x}(j) = q_{n,x}(j)\psi_n(j) + \psi_n(j + 1) \\
\psi_{n+1}(j) = \psi(j) + h_n(j - 1)\psi_n(j - 1)
\end{cases}
\]
Lax presentations for infinite lattice

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leads to the following Lax presentation for the Toda lattice:

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\psi_{n+1}(j) &= \psi(j) + h_n(j - 1)\psi_n(j - 1)
\end{align*}
\]

There is another Lax presentation:

\[
\begin{align*}
\varphi_{n,x}(j) &= \exp(q_{n+1}(j) - q_{n+1}(j - 1))\varphi_n(j - 1) - \varphi_n(j) \\
\varphi_{n-1}(j) &= -\exp(q_n(j) - q_{n+1}(j))\varphi_n(j) + \varphi_n(j + 1)
\end{align*}
\]
Involutions in the semidiscrete case

- In the continuous case the Toda lattice admits two types of involutions: reflection about an integer point (e.g. $h(-j) = h(j)$ that corresponds to $C$-series) and reflection about a half-integer point (e.g. $h(-j) = h(j + 1)$ that corresponds to $B$-series).
Involutions in the semidiscrete case

- In the continuous case the Toda lattice admits two types of involutions: reflection about an integer point (e.g. \( h(-j) = h(j) \) that corresponds to \( C \)-series) and reflection about a half-integer point (e.g. \( h(-j) = h(j+1) \) that corresponds to \( B \)-series).

- In the semidiscrete case there are no reflections about half-integer points! Involution \( h_n(-j) \rightarrow h_{n+j-c}(j-d) \) defines a reduction of the Toda lattice only if \( c = -2d \). The choice \( c = -1 \) leads to the boundary condition \( h_n(-2) = h_{n+1}(0) \) corresponding to \( C \)-series. Rewrite it as follows:

\[
q_n(-1) - q_{n-1}(-2) = q_{n+1}(1) - q_n(0).
\]
In continuum limit the latter equation tends to the $C$-series boundary condition.
In continuum limit the latter equation tends to the $C$-series boundary condition.

It looks as if there’s no semidiscrete analog to the $B$-series Toda lattice!
Remark

The $C$-series semidiscrete Toda lattice has an $n$-integral

$$\mu(x) = q_{n,x}(-1) + q_{n,x}(0) - h_n(0).$$

Although the condition $\mu = 0$ is not equivalent to the condition $h_n(-2) = h_{n+1}(0)$, these conditions are “almost equivalent”. Therefore further we’ll consider $\mu = 0$ as a boundary condition for the $C$-series lattice instead of the boundary condition $h_n(-2) = h_{n+1}(0)$. 

Sergey V. Smirnov
Semidiscrete Toda lattices
Lax pair for the $C$-series lattice

For the $C$-series semidiscrete Toda lattice Laplace invariants of hyperbolic equations corresponding to the above Lax pairs are related as follows:

\[
\begin{align*}
    h^\varphi(0) &= h(-1) = h(1) = h^\psi(1), & k^\varphi(0) &= h(0) = k^\psi(1), \\
    k^\psi(0) &= h(-1) = h(1) = k^\varphi(1), & h^\psi(0) &= h(0) = h^\varphi(1).
\end{align*}
\]
Lax pair for the $C$-series lattice

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$$h^\varphi(0) = h(-1) = h(1) = h^\psi(1), \quad k^\varphi(0) = h(0) = k^\psi(1),$$
$$k^\psi(0) = h(-1) = h(1) = k^\varphi(1), \quad h^\psi(0) = h(0) = h^\varphi(1).$$

Hence, there exist the functions $R = R_n(x)$ and $S = S_n(x)$ such that

$$\varphi_n(-1) = R_n\psi_{n+1}(0), \quad \psi_n(-1) = S_n\varphi_n(0).$$
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Hence, there exist the functions $R = R_n(x)$ and $S = S_n(x)$ such that

\[ \varphi_n(-1) = R_n \psi_{n+1}(0), \quad \psi_n(-1) = S_n \varphi_n(0). \]

This leads to the following Lax presentation for the $C$-series Toda lattice:
Lemma

The C-series semidiscrete Toda lattice is equivalent to the compatibility conditions for the following linear system:

\[
\partial_x(\psi) = A\psi, \quad T(\psi) = B\psi + L\Phi, \\
\partial_x(\Phi) = M\psi + C\Phi, \quad T^{-1}(\Phi) = D\Phi,
\]

where
Lax pairs in the continuous case

Symmetry approach in the continuous case

Lax pairs in the semidiscrete case

Symmetry approach in the semidiscrete case

\[ A = \begin{pmatrix}
  p_n & 0 & 0 & \ldots & 0 & 0 \\
  a_{n+1}(1)e^{-x} & -1 & 0 & \ldots & 0 & 0 \\
  0 & a_{n+1}(2) & -1 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & -1 & 0 \\
  0 & 0 & 0 & \ldots & a_{n+1}(r) & -1 \\
\end{pmatrix}, \]

\[ B = \begin{pmatrix}
  b_n(0) & e^x & 0 & 0 & \ldots & 0 \\
  0 & b_n(1) & -1 & 0 & \ldots & 0 \\
  0 & 0 & b_n(2) & -1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \ldots & -1 \\
  0 & 0 & 0 & 0 & \ldots & b_n(r) \\
\end{pmatrix}, \]
Outline
Introduction
Lax pairs in the continuous case
Symmetry approach in the continuous case
Lax pairs in the semidiscrete case
Symmetry approach in the semidiscrete case

\[ C = \begin{pmatrix}
q_{n,x}(0) & 1 & 0 & \ldots & 0 \\
0 & q_{n,x}(1) & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & q_{n,x}(r) & 0 \\
\end{pmatrix}, \]

\[ D = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
h_n(0) & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & h_n(r-1) & 1 \\
\end{pmatrix}, \]
L = \begin{pmatrix} f_n & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad M = \begin{pmatrix} g_n & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}

a_n(j) = \exp(q_n(j) - q_n(j - 1)), \quad b_n(j) = -\exp(q_n(j) - q_{n+1}(j)),

f_n = (-1)^n \exp(-q_n(-1)), \quad g_n = (-1)^n \exp(q_{n+1}(0)),

p_n = u_{n,x}(-1) + u_{n+1,x}(0).
Non-local variables

In the semi-discrete case symmetries are also expressed in terms of non-local variables.
Non-local variables

In the semi-discrete case symmetries are also expressed in terms of non-local variables. Non-localities $b_n^{(1)}(j)$ are defined as follows:

$$\partial_x b_n(j) = b_n^{(1)}(j) - b_n^{(1)}(j - 1), \quad \partial_n b_n^{(1)}(j) = \partial_x h_n(j).$$

Consistency of these equations is provided by the Toda lattice.
Further non-localities are defined by the following

**Lemma**

Non-local variables \( b_n^{(k)}(j) \), where \( k = 2, 3, \ldots \), satisfy the following relations:

\[
\begin{align*}
\partial_n b_n^{(k)}(j) &= h_n(j) b_{n+1}^{(k-1)}(j + 1) - h_{n+k-1}(j + k - 1) b_{n+1}^{(k-1)}(j) \\
\partial_x b_n^{(k)}(j) &= b_n^{(k)}(j) (b_{n+k}(j + k) - b_n(j)) + \\
&\quad + h_{n+k-1}(j + k - 1) \left( b_n^{(k)}(j - 1) - b_n^{(k)}(j) \right) + \\
&\quad + b_n^{(k+1)}(j) - b_n^{(k+1)}(j - 1)
\end{align*}
\]

Consistency of these equations for any \( k = 2, 3, \ldots \) is provided by the first of them for \( k + 1 \).
Symmetries of the infinite lattice

The infinite semidiscrete Toda lattice admits the following symmetries:

- second order symmetry

\[ q_{n,t}(j) = b_n^2(j) + b_n^{(1)}(j) + b_n^{(1)}(j - 1); \]
Symmetries of the infinite lattice

The infinite semidiscrete Toda lattice admits the following symmetries:

- second order symmetry

\[ q_{n,t}(j) = b_n^2(j) + b_n^{(1)}(j) + b_n^{(1)}(j - 1); \]

- third order symmetry

\[
q_{n,t}(j) = \begin{array}{c}
q_n^3(j) + b_n^{(2)}(j) + b_n^{(2)}(j - 1) + b_n^{(2)}(j - 2) + \\
+ b_n^{(1)}(j) (2b_n(j) + b_n(j + 1)) + \\
= b_n^{(1)}(j - 1) (2b_n(j) + b_n(j - 1)) - \\
- b_n^{(1)}(j) h_n(j + 1) - b_n^{(1)}(j - 1) h_n(j) - b_n^{(1)}(j - 2) h_n(j - 1)
\end{array}
\]
New set of variables

In the new set of variables

\[ u_n = e^{-q_n(-2)}, \quad v_n = e^{q_{n+1}(-1)}, \quad w_n = q^{-q_n(0)}, \quad z_n = e^{q_{n+1}(1)} \]

symmetries are expressed in a more simple way:

\[
\begin{align*}
    u_{n,t} &= u_{n,xxx} + 3r_n u_{n,x} - 3s_n u_n + 3r_{n,x} u_n, \\
    v_{n,t} &= v_{n,xxx} + 3r_{n+1} v_{n,x} + 3s_{n+1} v_n \\
    w_{n,t} &= w_{n,xxx} + 3\rho_n w_{n,x} - 3\sigma_n w_n + 3\rho_{n,x} w_n, \\
    z_{n,t} &= z_{n,xxx} + 3\rho_{n+1} z_{n,x} + 3\sigma_{n+1} z_n.
\end{align*}
\]

where

\[
\begin{align*}
    r_n &= b_n^{(1)}(-2), \quad s_n = b_n^{(2)}(-2) + r_n(b_n(-1) - h_n(-1)), \\
    \rho_n &= b_n^{(1)}(0), \quad \sigma_n = b_n^{(2)}(0) + \rho_n(b_n(1) - h_n(1)).
\end{align*}
\]
Classification theorem

What boundary conditions are compatible with the above symmetries?
Classification theorem

What boundary conditions are compatible with the above symmetries?

- Only trivial boundary condition $u = 0$ is compatible with second order symmetry.
Classification theorem

What boundary conditions are compatible with the above symmetries?

- Only trivial boundary condition $u = 0$ is compatible with second order symmetry.

- If the variables

$$v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}, v_{n-1,x}, w_{n,x}, z_{n,x}$$

are independent, then only trivial boundary condition $u = 0$ is compatible with the third order symmetry ($A$ series).
If $v_{n-1,x} = H(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}, w_{n,x}, z_{n,x})$ and boundary condition $u = F(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1})$ is compatible with the third order symmetry, then

$$H = \frac{w_{n,x}}{w_n} v_{n-1} + z_n w_n v_{n-1}$$

and

$$F = \frac{w_{n+1} z_{n+1}}{v_n} \quad (C \text{ series}).$$
If \( v_{n-1,x} = H(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}, w_{n,x}, z_{n,x}) \) and boundary condition \( u = F(v_{n-1}, v_n, w_n, w_{n+1}, z_n, z_{n+1}) \) is compatible with the third order symmetry, then

\[
H = \frac{w_{n,x}}{w_n} v_{n-1} + z_n w_n v_{n-1}
\]

and

\[
F = \frac{w_{n+1} z_{n+1}}{v_n} \quad (C \text{ series}).
\]

Note that the equation for the function \( H \) is equivalent to the “modified” \( C \)-series boundary condition

\[
q_{n,x}(-1) + q_{n,x}(0) - h_n(0) = 0.
\]
Thank you!