Building Abelian Functions with Generalised Hirota Operators

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Motivation

Abelian functions
The basis problem

Generalised Hirota operators and \( R \)-functions

Definitions
Proofs and examples
1 Motivation
   - Abelian functions
   - The basis problem

2 Generalised Hirota operators and $R$-functions
   - Definitions
   - Proofs and examples
Abelian functions associated to algebraic curves

We consider functions periodic w.r.t. the period lattice $\Lambda$ of an algebraic curve of genus $g$. If $\omega_1, \omega_2$ are the period matrices then

$$\Lambda = \{ \omega_1 m + \omega_2 n \mid m, n \in \mathbb{Z}^g \}.$$  

**Definition**

Let $M(u)$ be a meromorphic function of $(u_1, u_2, \ldots, u_g) = u \in \mathbb{C}^g$. Then $M(u)$ is an **Abelian function associated with the curve** if

$$M(u + \omega_1 n + \omega_2 m) = M(u),$$

for all integer vectors $n, m \in \mathbb{Z}^g$ where $M(u)$ is defined.

The simplest case are the elliptic functions, when $g = 1$. 

Matthew England  
Hirota Operators and Abelian Functions
Kleinian $\wp$-functions

Define the Kleinian $\wp$-functions as the second log derivatives of the curve’s multivariate $\sigma$-function, $\sigma = \sigma(u) = \sigma(u_1, u_2, ..., u_g)$:

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u), \quad i \leq j \in \{1, 2, ..., g\}.$$ 

We can extend this notation to higher order derivatives:

$$\wp_{i_1, ..., i_m} = -\frac{\partial^m}{\partial u_{i_1} \ldots \partial u_{i_m}} \ln \sigma(u), \quad i_1 \leq \cdots \leq i_m \in \{1, 2, ..., g\}.$$ 

They are all Abelian functions and those with $m$ indices have poles of order $m$ on the $\Theta$-divisor.
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Abelian functions also finding applications.

- Solutions to various integrable equations from the KP-hierarchy.
- Also used to describe the Kummer surface, to describe the motion of a double pendulum, and to give the geodesic motions in certain space-time metrics.
Let $\Gamma(m)$ be the vector space of Abelian functions with poles of order at most $m$ occurring only on the $\Theta$-divisor. We seek bases for such spaces.

- The Riemann-Roch theorem for Abelian varieties gives $\text{dim}(\Gamma(m)) = g^m$.
- We start by including the basis for $\Gamma(m - 1)$. We then need to find $g^m - g^{m-1}$ linearly independent functions with poles of order exactly $m$.
- The $m$-index Kleinian $\wp$-functions may all be included.
### Example: Genus one case

A table of bases for the elliptic case: Here $\wp$ is the Weierstrass elliptic $\wp$-function.
Example: Genus two case

<table>
<thead>
<tr>
<th>Space</th>
<th>Dim</th>
<th>Basis for $\Gamma(m) \setminus \Gamma(m - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma(0)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma(1)$</td>
<td>1</td>
<td>$\wp_{11}, \wp_{12}, \wp_{22}$</td>
</tr>
<tr>
<td>$\Gamma(2)$</td>
<td>4</td>
<td>$\wp_{111}, \wp_{112}, \wp_{122}, \wp_{222}, \Delta$</td>
</tr>
<tr>
<td>$\Gamma(3)$</td>
<td>9</td>
<td>$\wp_{1111}, \ldots, \wp_{2222}, \partial_1 \Delta, \partial_2 \Delta$</td>
</tr>
<tr>
<td>$\Gamma(4)$</td>
<td>16</td>
<td>$\wp_{11111}, \ldots, \wp_{22222}, \partial_{11} \Delta, \partial_{12} \Delta, \partial_{22} \Delta$</td>
</tr>
<tr>
<td>$\Gamma(5)$</td>
<td>25</td>
<td>$\wp_{111111}, \ldots, \wp_{222222}, \partial_{111} \Delta, \partial_{112} \Delta, \partial_{122} \Delta$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\Gamma(m)$</td>
<td>$m^2$</td>
<td>${\wp_{i_1 \ldots i_m}}, {\partial_{i_1 \ldots \partial_{i_m-2}} \Delta}$</td>
</tr>
</tbody>
</table>

A table of bases for the genus two case: Here we use genus two Kleinian $\wp$-functions and the function $\Delta = \wp_{11} \wp_{22} - \wp_{12}^2$ along with its derivatives. We use $\partial_i$ for differentiation with respect to $u_i$ and $\{\cdot\}$ to denote all functions of that form.
We know the dimension of $\Gamma(m)$. For $(n, s)$-curves at least, there are well developed techniques to test the linear independence of elements using the $\sigma$-expansion and weight arguments. The **basis problem** is hence the identification of enough suitable functions of a given pole order.
The basis problem

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- Bases have been built for various specific cases. These required new functions, usually defined as polynomials in $\wp$-functions with coefficients identified through a pole matching procedure, analogous to $\Delta$. 
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- We know the dimension of $\Gamma(m)$. For $(n, s)$-curves at least, there are well developed techniques to test the linear independence of elements using the $\sigma$-expansion and weight arguments. The basis problem is hence the identification of enough suitable functions of a given pole order.

- Bases have been built for various specific cases. These required new functions, usually defined as polynomials in $\wp$-functions with coefficients identified through a pole matching procedure, analogous to $\Delta$.

- However, for $g > 2$ hyperelliptic and $g > 3$ non-hyperelliptic, these bases are not finitely generated by differentiation, so infinitely many new functions are required.
We can define Hirota’s bilinear operator as

$$\mathcal{D}_i = \frac{\partial}{\partial u_i} - \frac{\partial}{\partial v_i}.$$ 

We may then check that

$$\wp_{ij}(u) = \frac{(-1)}{2\sigma(u)^2} \mathcal{D}_i\mathcal{D}_j\sigma(u)\sigma(v) \bigg|_{v=u}.$$
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$$\wp_{ij}(u) = \frac{(-1)}{2\sigma(u)^2} D_i D_j \sigma(u) \sigma(v) \bigg|_{v=u}.$$ 

This was identified by Baker and so a more appropriate name for $D_i$ might be a Baker-Hirota operator.
We define the \( n \)-index \( Q \)-functions as

\[
Q_{i_1,i_2,\ldots,i_n}(u) = \frac{(-1)}{2\sigma(u)^2} D_{i_1} D_{i_2} \cdots D_{i_n} \sigma(u) \sigma(v) \bigg|_{v=u}
\]

- So \( Q_{ij} = \wp_{ij} \). But further indices denote application of operators, not differentiation.
- Note that is applied with \( n \) odd then they are identically zero.
- First \( Q \)-function used by Baker. The 4-index functions defined in EEMOP07. Six index functions required for theory of non-hyperelliptic genus six curves.
All $Q$-functions are Abelian functions with poles of order two and so can be used to construct bases of $\Gamma(2)$.

- **Genus 3 hyperelliptic**: $\{1, \wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}, Q_{2222}\}$.
- **Genus 3 trigonal**: $\{1, \wp_{11}, \wp_{12}, \wp_{13}, \wp_{22}, \wp_{23}, \wp_{33}, Q_{1333}\}$.
- **Genus 6 tetragonal**: $\{\wp_{11}, \ldots, \wp_{66}, Q_{5566}, \ldots, Q_{1144}, Q_{114466}\}$.
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- **Genus 6 tetragonal:** $\{\wp_{11}, \ldots, \wp_{66}, Q_{5566}, \ldots, Q_{1144}, Q_{114466}\}$.

The choice of functions is not unique. Usually we can replace a given function by any other that has the same weight. 

**Aim:** To define something similar to $Q$-functions, which can be used to complete bases for $\Gamma(m)$ where $m > 2$. 

Outline

1 Motivation
   - Abelian functions
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2 Generalised Hirota operators and $R$-functions
   - Definitions
   - Proofs and examples
The original Hirota derivatives act on pairs, not products.

The Hirota operator is bilinear over addition and scalar multiplication, so we can treat the pairs as tensor products. The original Hirota derivative is then a composition of the Hirota operators with a total symmetrization operator, $S$, mapping from the tensor to the differential polynomial algebra:

$$F \otimes F \xrightarrow{D_i} F \otimes F \xrightarrow{S} F,$$

$$f \otimes g \xrightarrow{D_i} f_i \otimes g - f \otimes g_i \xrightarrow{S} f_i g - f g_i.$$  

where $F$ denotes a $k$-algebra of appropriate $k$-valued functions, for some field $k$. In this talk we let $k = \mathbb{C}$. 
We now work with the tensor product of \( m \) multivariate functions.

\[
\bigotimes_{k=1}^{m} f[k] = f[1] \otimes \ldots \otimes f[m].
\]

The variables are \( u = (u_1, \ldots, u_g) \). Define the operator \( \partial_i^{[j]} \) by

\[
\partial_i^{[j]} (f[1] \otimes \ldots \otimes f[j] \otimes \ldots \otimes f[m]) = f[1] \otimes \ldots \otimes f_i^{[j]} \otimes \ldots \otimes f[m].
\]
Generalised Hirota operators

We now work with the tensor product of $m$ multivariate functions.

$$\bigotimes_{k=1}^m f[k] = f[1] \otimes \ldots \otimes f[m].$$

The variables are $u = (u_1, \ldots, u_g)$. Define the operator $\partial_i^{[j]}$ by

$$\partial_i^{[j]} (f[1] \otimes \ldots \otimes f[i] \otimes \ldots \otimes f[m]) = f[1] \otimes \ldots \otimes f_i^{[j]} \otimes \ldots \otimes f[m].$$

Generalised Hirota operators

Define $\mathcal{H}_i^{[m]}$ to be the $m$th order generalised Hirota operator which acts on the tensor product of $m$ functions as

$$\mathcal{H}_i^{[m]} \left( \bigotimes_{k=1}^m f[k] \right) = \sum_{j=1}^m \zeta^{j-1} \partial_i^{[j]} \bigotimes_{k=1}^m f[k],$$

where $\zeta$ is a primitive $m$th root of unity.
When \( m = 2 \) we have

\[
\mathcal{H}^{[2]}_i (f \otimes g) = f_i \otimes g - f \otimes g_i
\]

The total symmetrization operator acts as

\[
S \left( \bigotimes_{k=1}^{m} f[k] \right) = \prod_{k=1}^{m} f[k].
\]

So we see,

\[
Q_{i_1, \ldots, i_n} = -\frac{1}{2\sigma^2} S \circ \mathcal{H}^{[2]}_{i_1} \circ \cdots \circ \mathcal{H}^{[2]}_{i_n} (\sigma \otimes \sigma).
\]
Define the $n$-index $m$th order $\mathcal{R}$-functions as

$$\mathcal{R}_{i_1, \ldots, i_n}^{[m]} = \left( -\frac{1}{m \sigma^m} \right) \left( S \circ \mathcal{H}_{i_1}^{[m]} \circ \ldots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} \sigma \right).$$
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\]

- The functions clearly have poles of order $m$. We find they are also Abelian and so can be used to construct bases of $\Gamma(m)$.
- We find that they are identically zero unless $m|n$.
- We have,
\[
Q_{i_1,\ldots,i_n} = \mathcal{R}_{i_1,\ldots,i_n}^{[2]}
\]
\[
\wp_{i_1,\ldots,i_n} = \mathcal{R}_{i_1,\ldots,i_n}^{[n]}
\]
The $\sigma$-function is quasi-periodic. For $\ell \in \Lambda$ we have 

$$\sigma(u + \ell) = h\sigma(u) \text{ where } h = \chi e^{L(u,\ell)} \text{ and } L \text{ is s.t. all } L_{ij} = 0.$$
Sketch proof of periodicity I

1. The $\sigma$-function is quasi-periodic. For $\ell \in \Lambda$ we have $\sigma(u + \ell) = h\sigma(u)$ where $h = \chi e^{L(u,\ell)}$ and $L$ is s.t. all $L_{ij} = 0$.

2. Define a multiplication operation between tensor products as

$$\bigotimes_{k=1}^{m} f[k] \cdot \bigotimes_{k=1}^{m} g[k] = \bigotimes_{k=1}^{m} f[k] g[k].$$
Sketch proof of periodicity 1

1. The $\sigma$-function is quasi-periodic. For $\ell \in \Lambda$ we have $\sigma(u + \ell) = h \sigma(u)$ where $h = \chi e^{L(u,\ell)}$ and $L$ is s.t. all $L_{ij} = 0$.

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3. The Hirota operators satisfy a corresponding product rule,

$$\mathcal{H}_i^{[m]} \left( \bigotimes_{k=1}^{m} f[k] \cdot \bigotimes_{k=1}^{m} g[k] \right) = \bigotimes_{k=1}^{m} f[k] \cdot \mathcal{H}_i^{[m]} \left( \bigotimes_{k=1}^{m} g[k] \right) + \bigotimes_{k=1}^{m} g[k] \cdot \mathcal{H}_i^{[m]} \left( \bigotimes_{k=1}^{m} f[k] \right).$$
There is a corresponding Leibniz Rule,

\[ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \left( \bigotimes_{k=1}^{m} f^{[k]} \right) \cdot \left( \bigotimes_{k=1}^{m} g^{[k]} \right) = \sum_{\ell=0}^{n} \sum_{\pi \in \Pi} \mathcal{H}_{\pi_1}^{[m]} \left( \bigotimes_{k=1}^{m} f^{[k]} \right) \cdot \mathcal{H}_{\pi_2}^{[m]} \left( \bigotimes_{k=1}^{m} g^{[k]} \right). \]

\( \Pi \) the set of disjoint partitions, \( \pi \), of the indices \( \{i_1, \ldots, i_n\} \) into two subsets \( \pi_1 \) and \( \pi_2 \) of lengths \( n - \ell \) and \( \ell \). \( \mathcal{H}_{\pi_i}^{[m]} \) denotes concatenation of generalised Hirota operators.
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\[ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \left( \bigotimes_{k=1}^{m} f^{[k]} \right) \cdot \left( \bigotimes_{k=1}^{m} g^{[k]} \right) \]

\[ = \sum_{\ell=0}^{n} \sum_{\pi \in \Pi} \mathcal{H}_{\pi_1}^{[m]} \left( \bigotimes_{k=1}^{m} f^{[k]} \right) \cdot \mathcal{H}_{\pi_2}^{[m]} \left( \bigotimes_{k=1}^{m} g^{[k]} \right). \]

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We may show inductively that for all \( n \)

\[ S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} h = 0. \]
So putting it all together,

\[ R_{i_1, \ldots, i_n}^{[m]}(u + \ell) = \left( -\frac{1}{m(h\sigma)^m} \right) \left( S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} h\sigma \right) \] by 1

\[ = \left( -\frac{1}{mh^m\sigma^m} \right) \left( S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} h \bigotimes_{k=1}^{m} \sigma \right) \] by 2

\[ = \left( -\frac{1}{mh^m\sigma^m} \right) \left( S \circ \sum_{\ell=0}^{n} \sum_{\pi \in \Pi} \mathcal{H}_{\pi_1}^{[m]} \bigotimes_{k=1}^{m} h \bigotimes_{k=1}^{m} \sigma \right) \] by 4

\[ = \left( -\frac{1}{mh^m\sigma^m} \right) \left( h^m \cdot S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} \sigma \right) \] by 5

\[ = R_{i_1, \ldots, i_n}^{[m]}(u). \]
We observe that

\[ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \otimes f[k] = \sum_{\rho \in P_n^m} \sum_{\psi \in \Psi(\rho)} \zeta^{z(\psi)} \sum_{\pi \in \Pi(\psi)} \otimes_{k=1}^m f_{\pi_k} \]

where \( \rho \) are partitions of \( n \), \( \psi \) permutations of \( \rho \) and \( \pi \) set partitions of \( \{i_1, \ldots, i_n\} \) into subsets of lengths given by \( \psi \).
We observe that

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where \( \rho \) are partitions of \( n \), \( \psi \) permutations of \( \rho \) and \( \pi \) set partitions of \( \{i_1, \ldots, i_n\} \) into subsets of lengths given by \( \psi \).

Then, with \( X_k = \zeta^{k-1} \) we can show

\[ S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_n}^{[m]} \bigotimes_{k=1}^{m} f = \sum_{\rho \in P_n^m} M_\rho(X) \left( \sum_{\pi \in \Pi(\rho)} \left( \prod_{k=1}^{m} f_{\pi_k} \right) \right). \]

Here \( M_\rho(x) \) are monomial symmetric functions. E.g.

\[ M_{[2,1,0]}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_3^2 + x_1^2 x_3 + x_2^2 x_3 + x_2 x_3^2. \]
For $X_k = \zeta^{k-1}$ the power sum symmetric functions are

$$p_k = \begin{cases} 
  m & \text{if } m \mid k \\
  0 & \text{otherwise}
\end{cases}$$
For $X_k = \zeta^{k-1}$ the power sum symmetric functions are

$$p_k = \begin{cases} m & \text{if } m|k \\ 0 & \text{otherwise} \end{cases}$$

All symmetric functions may be expressed as a polynomial in $p_1, \ldots, p_m$. Using the degree of polynomials as a grading, we see that unless $m|n$, then such expressions for $M_\rho(x)$ will be zero, and hence $R_{i_1,\ldots,i_n}$ also.
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All symmetric functions may be expressed as a polynomial in $p_1, \ldots, p_m$. Using the degree of polynomials as a grading, we see that unless $m \mid n$, then such expressions for $M_\hat{\rho}(x)$ will be zero, and hence $\mathcal{R}^{[m]}_{i_1, \ldots, i_n}$ also.

If $n \mid m$ then $\hat{\rho} = [n, 0, \ldots, 0] \in P^m_n$ and $M_\hat{\rho}(X) = p_n(X) \neq 0$, so $\mathcal{R}^{[m]}_{i_1, \ldots, i_n}$ is not identically zero in these cases.
We introduce $\hat{g}$ such that $\sigma = e^{\hat{g}}$. Then $g_{i_1,...,i_n} = -\hat{g}_{i_1,...,i_n}$. 

Sketch proof that all Kleinian $g$-functions are $R$-functions
Motivation

Generalised Hirota operators and $R$-functions

Definitions

Proofs and examples

Sketch proof that all Kleinian $\wp$-functions are $R$-functions

1. We introduce $\hat{\wp}$ such that $\sigma = e^{\hat{\wp}}$. Then $\wp_{i_1, \ldots, i_n} = -\hat{\wp}_{i_1, \ldots, i_n}$.

2. So applying a Hirota operator:

$$
\mathcal{H}_{i_1}^{[m]} \bigotimes_{k=1}^{m} \sigma = \sum_{j=1}^{m} \zeta^{-1} \partial_{[j]}^{m} \bigotimes_{k=1}^{m} e^{\hat{\wp}}
$$

$$
= \sum_{j=1}^{m} \zeta^{-1} (1 \otimes \cdots \otimes \hat{\wp}_{i_1} \otimes \cdots \otimes 1) \left( \bigotimes_{k=1}^{m} e^{\hat{\wp}} \right) =: \Sigma_1 \left( \bigotimes_{k=1}^{m} \sigma \right)
$$
Sketch proof that all Kleinian $\wp$-functions are $R$-functions

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$$= \sum_{j=1}^{m} \zeta^{j-1} (1 \otimes \cdots \otimes \hat{\wp}_{i_1} \otimes \cdots \otimes 1) \left( \bigotimes_{k=1}^{m} e^{\hat{\wp}} \right) =: \Sigma_1 \left( \bigotimes_{k=1}^{m} \sigma \right)$$

3. Applying another Hirota operator to $\Sigma_1$ will give many terms, but all except one will evaluate to zero upon application of $S$.

$$\mathcal{H}_{i_2}^{[m]} (\Sigma_1) \triangleq \sum_{j=1}^{m} \zeta^{2(j-1)} (1 \otimes \cdots \otimes \hat{\wp}_{i_1 i_2} \otimes \cdots \otimes 1) =: \Sigma_2$$
Sketch proof that all Kleinian \( \wp \)-functions are \( R \)-functions

So we have

\[
\mathcal{H}_{i_1}^{[m]} \circ \mathcal{H}_{i_2}^{[m]} \bigotimes_{k=1}^m \sigma = \sum_2 \left( \bigotimes_{k=1}^m \sigma \right) + \sum_1 \mathcal{H}_{i_2}^{[m]} \left( \bigotimes_{k=1}^m \sigma \right).
\]
Sketch proof that all Kleinian $\wp$-functions are $R$-functions

So we have

$$\mathcal{H}_{i_1}^{[m]} \circ \mathcal{H}_{i_2}^{[m]} \bigotimes_{k=1}^{m} \sigma = \Sigma_2 \left( \bigotimes_{k=1}^{m} \sigma \right) + \Sigma_1 \mathcal{H}_{i_2}^{[m]} \left( \bigotimes_{k=1}^{m} \sigma \right).$$

In general,

$$\mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_m}^{[m]} \bigotimes_{k=1}^{m} \sigma = \Sigma_m \left( \bigotimes_{k=1}^{m} \sigma \right) + \ldots$$

where

$$\Sigma_m = \sum_{j=1}^{m} \zeta^{m(j-1)} \left( 1 \otimes \cdots \otimes \hat{\wp}_{i_1, \ldots, i_m} \otimes 1 \otimes 1 \right)$$

and all other terms, have as a factor, a sequence of $\mathcal{H}$ of size less than $m$, applied to a tensor product of $\sigma$. Hence under $S$,

$$S \circ \mathcal{H}_{i_1}^{[m]} \circ \cdots \circ \mathcal{H}_{i_m}^{[m]} \bigotimes_{k=1}^{m} \sigma = m \hat{\wp}_{i_1, \ldots, i_m} \sigma^m.$$

So $R_{i_1, i_2, \ldots, i_m}(u) = \wp_{i_1, i_2, \ldots, i_m}(u)$. 
Example: Bases for hyperelliptic genus 3 curve: (2,7)-curve

$$\Gamma(2):$$

$$\{1, R_{11}^2, R_{12}^2, R_{13}^2, R_{22}^2, R_{23}^2, R_{33}^2, R_{2222}^2\}.$$
Example: Bases for hyperelliptic genus 3 curve: (2,7)-curve

\( \Gamma(2) : \)
\[ \{ 1, R_{11}^2, R_{12}^2, R_{13}^2, R_{22}^2, R_{23}^2, R_{33}^2, R_{2222}^2 \} \].

\( \Gamma(3) \setminus \Gamma(2) : \)
\[ \left\{ R_{111}^3, R_{122}^3, R_{222}^3, R_{333}^3, R_{233333}^3, R_{222222}^3, \partial_1 R_{2222}^2, \partial_2 R_{2222}^2, \partial_3 R_{2222}^2 \right\}. \]

\( \Gamma(4) \setminus \Gamma(3) : \)
\[ \left\{ R_{1111}^4, R_{1133}^4, R_{2222}^4, \partial_3 R_{233333}^3, \partial_1 R_{133333}^3, \partial_2 R_{123333}^3, \partial_2 R_{112333}^3, \partial_1 R_{111233}^3, \partial_2 R_{111333}^3, \partial_3 R_{112233}^3, \partial_3 R_{111333}^3, \partial_1 R_{111233}^3, \partial_3 R_{111333}^3, \partial_1 R_{111233}^3, \partial_3 R_{111233}^3 \right\}. \]
Example: Bases for trigonal genus 3 curve: $(3,4)$-curve

$\Gamma(2)$:

$\{1, R_{11}, R_{12}, R_{13}, R_{22}, R_{23}, R_{33}, R_{2222}\}$.

$\Gamma(3) \setminus \Gamma(2)$:

$$\left\{ R_{111}^{[3]}, R_{122}^{[3]}, R_{222}^{[3]}, R_{333}^{[3]}, R_{33333}^{[3]}, R_{11333}^{[3]}, \partial_1 R_{2222}^{[2]}, \right.$$ \hfill $$
\left. R_{1222}^{[3]}, R_{1233}^{[3]}, R_{2233}^{[3]}, R_{22223}^{[3]}, R_{12233}^{[3]}, \partial_2 R_{2222}^{[2]}, \partial_3 R_{2222}^{[2]} \right\}.$$

$\Gamma(4) \setminus \Gamma(3)$:

$$\left\{ R_{1111}^{[4]}, R_{1133}^{[4]}, R_{2222}^{[4]}, \partial_3 R_{33333}^{[3]}, \partial_2 R_{13333}^{[3]}, \partial_2 R_{12233}^{[3]}, \partial_2 R_{12223}^{[3]}, R_{223333}^{[4]}, R_{222333}^{[4]}, R_{223333}^{[4]}, \partial_4 R_{222222}^{[4]}, \partial_4 R_{233333}^{[4]}, \partial_4 R_{222333}^{[4]}, \partial_4 R_{223333}^{[4]}, \partial_4 R_{222233}^{[4]}, \partial_4 R_{222223}^{[4]}, \partial_4 R_{222222}^{[4]} \right\}.$$
The two genus three cases have *structurally similar bases*. I.e. The same number of functions of each *type* used in corresponding bases. Not apparent when same bases constructed in EEO11.

**Conjecture 1:** Bases of Abelian functions associated with curves of the same genus share more structure than previously thought.
Conjectures arising from examples

The two genus three cases have *structurally similar bases*. I.e. The same number of functions of each *type* used in corresponding bases. Not apparent when same bases constructed in EEO11.

**Conjecture 1:** Bases of Abelian functions associated with curves of the same genus share more structure than previously thought.

It appears that $\Gamma(m) \backslash \Gamma(m - 1)$ may always be filled using $m$th order $\mathcal{R}$-functions and first derivatives of $(m - 1)$th order $\mathcal{R}$-functions.

**Conjecture 2:** The standard Abelian functions are spanned by $\mathcal{R}$ and $\partial \mathcal{R}$. 
Further Information

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*Building Abelian functions with generalised Hirota operators.*

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Thanks for listening.
The functions associated to a given curve satisfy an addition formula of the form

\[
\frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = \sum_i c_i A_i(u) B_i(v)
\]

where \(A_i, B_i \in \Gamma(2)\) and the \(c_i\) are constants.

- Determine exact formula using bases and \(\sigma\)-expansion.
- Use available simplifications; symmetry, parity, weights.
- Generalises classic Weierstrass formula,

\[
\frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u).
\]
Another class of addition formulae exist for cyclic \((n, s)\)-curves,

\[ y^n = x^s + \lambda_{s-1}x^{s-1} + \ldots + \lambda_1x + \lambda_0 \]

associated with their family of automorphisms:

\[ [\zeta^j]: (x, y) \rightarrow (x, \zeta^jy), \quad \text{where } \zeta^n = 1. \]
Automorphism addition formulae

Another class of addition formulae exist for cyclic \((n, s)\)-curves,

\[ y^n = x^s + \lambda_{s-1}x^{s-1} + ... + \lambda_1x + \lambda_0 \]

associated with their family of automorphisms:

\[ [\zeta^j] : (x, y) \rightarrow (x, \zeta^jy), \quad \text{where } \zeta^n = 1. \]

The following function will be Abelian in \(u[i], i = 1 \ldots n:\)

\[ \prod_{j=1}^{n} \sigma \left( \frac{\sigma \left( \sum_{i=1}^{n} [\zeta^{i+j}]u[i] \right)}{\sigma(u[j])^n} \right) \]

So we can write is as sum of products of \(n\) functions from \(\Gamma(n)\) in the variables, \(u[i], i = 1 \ldots n\) respectively.
Simplified and reduced addition formulae

Such formulae can be difficult to compute. Simplified versions may be found, setting one or more of the variables $u^{[i]}$ to zero.
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We can also consider reduced curves which have further automorphisms and hence extra addition formulae.

**Example:** The restricted (3,4)-curve, $y^3 = x^4 + \lambda_0$ has automorphisms

$$[i^j] : (x, y) \mapsto ((i)^j x, y), \quad \text{where } i = \sqrt{-1}.$$
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$$[i^j]: (x, y) \mapsto ((i^j)x, y), \text{ where } i = \sqrt{-1}.$$

The functions associated to this curve satisfy

$$\frac{\sigma(u + v)\sigma(u + [i]v)\sigma(u + [i^2]v)\sigma(u + [i^3]v)}{\sigma(u)^4\sigma(v)^4} = f(u, v) - f(v, u)$$

where $f$ is a quadratic in functions of $\Gamma(4)$.