Locally Boolean Domains

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Abstract

Bistable bidomains have been used to give a simple order-theoretic construction of a cartesian closed category of sequential functions. In this paper, we investigate the intensional properties of a full subcategory, the *locally boolean domains*, in which the bistable structure is given by an involution operation. We show that every pointed locally boolean domain is the limit of an ω-chain of "prenex normal forms" constructed using only products and lifted sums. We use this result to describe a model of linear logic (incorporating both intuitionistic and polarized classical fragments). We show that affine and bistable functions correspond to unique "strategies" on the associated normal forms, and that function composition corresponds to "parallel composition plus hiding" of these strategies.

1 Introduction

A longstanding problem in domain theory has been to find a simple characterization of higher-order sequential functions which is wholly extensional in character. Typically, what is sought is some form of mathematical structure, such that functions which preserve this structure are sequential *and* which can be used to construct a cartesian closed category.

In previous work [11,10], we gave such an extensional description of Cartwright and Felleisen's *observably sequential functionals* [4], based on the notion of *bistable biorder* and bistable function.

Definition 1.1 *A* bistable biorder *is a tuple* $(|D|, \sqsubseteq_D, \updownarrow_D)$ *where* $(|D|, \sqsubseteq)$ *is a partial order, and* $\updownarrow$ *is an equivalence relation ("bistable coherence")such that for each* $x \in |D|$*, the equivalence class of* x *is a* distributive lattice *and the inclusion of* $[x]$ *into* $(|D|, \sqsubseteq)$ *preserves binary meets and joins.*

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A $\sqsubseteq$-monotone function between bistable biorders is bistable *if its restriction to every $\updownarrow$-equivalence class is a lattice homomorphism.*

Essentially, two programs are bistably coherent if they behave in the same way except in respect of their failures; where one diverges, the other may produce an untrappable error, $\top$. By requiring the preservation of bistably coherent meets and joins, bistability forces functions to be observably sequential. Moreover the category of bistable biorders and bistable functions is Cartesian closed, and the corresponding interpretation of the simply-typed λ-calculus with constants $\top$ and $\bot$ is *universal* — i.e. every element of a type-object is the denotation of a term.

In other words, from each bistable function, we can extract an algorithm for computing it — a λ-term. However, this syntactic characterisation of the "intensional content" of bistable functions is not fully satisfactory for a number of reasons. The term extracted for each function is not unique, for example. And whilst it is technically possible to use universality to relate the bistable semantics to other universal models of this language, in practice this is unilluminating. Nor is it clear how reduction of terms (i.e. sequential computation) corresponds to composition of bistable functions. Thus the problem addressed by this paper is (in general terms) to give a more complete characterisation of the "intensional content" of bistable functions which does not have these limitations.

One aspect our account is to describe a "linear decomposition" of a CCC of bistable functions into a model of linear type-theory (although this is not fully classical, it does have a duality property corresponding to the Player/Opponent duality of games). We also aim to facilitate study of the relationship between the bistable model and intensional descriptions of the observably sequential functionals, of which there is a multiplicity, including Cartwright, Curien and Felleisen's sequential algorithms model [5], a game semantic presentation due to Lamarche [12,6], and Hyland and Schalke's graph games. This correspondence has been the object of some investigation [14,7], but the approach we take here is to relate bistable functions to a more abstract, order-theoretic notion of intensional behaviour.

We identify a subclass of bistable biorders in which each element has a "complement" obtained by swapping the roles of $\bot$ and $\top$ (and thus each $\updownarrow$-equivalence class is a boolean algebra). In fact, these *locally boolean orders* may be specified by giving only the extensional order and an involution operation. By imposing a simple algebraicity condition, to arrive at the notion of *locally boolean domain*, we can prove a rather strong representation result, which underpins this paper. We show that every such domain is a limit of a chain of "prenex normal forms" constructed using only (alternating) products and sums. This establishes the connection with intensional models; we may view normal forms directly as games, (or, indeed, sequential data structures), in which Opponent moves by choosing indices in the products and Player by choosing indices in the sum. Moreover, we show how each bistable and affine function determines a strategy on the associated game, and how composition of functions may be computed as the "parallel composition plus hiding" of the associated strategies.

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2 Locally Boolean Domains

Definition 2.1 *An* involution *for a partial order* $(P, \sqsubseteq)$ *is an endofunction* $\neg : P \to P$ *such that* $\neg\neg x = x$ *and if* $x \sqsubseteq y$*, then* $\neg y \sqsubseteq \neg x$*.*
A locally boolean order $A = (|A|, \sqsubseteq, \neg)$ *is a partial order with an involution such that:*

- *for each* x*,* $\{x, \neg x\}$ *has a least upper bound, which we shall write* $x^\top$ *(and thus* $\{x, \neg x\}$ *has a greatest lower bound* $x_\bot = \neg(x^\top)$*).*
- *We shall write* $x \uparrow y$ *if* $x \sqsubseteq y^\top$ *and* $y \sqsubseteq x^\top$ *and require that if* $x \uparrow y$*, then* $\{x, y\}$ *has a greatest lower bound* $x \wedge y$ *and a least upper bound* $x \vee y$*.*

A is complete *if* $(|A|, \sqsubseteq)$ *is a cpo — i.e. every* $\sqsubseteq$*-directed set* X *has a least upper bound* $\bigsqcup X$*. A is* pointed *if the empty set has a least upper bound — i.e. A has a least element* $\bot$ *and hence a greatest element* $\top = \neg\bot$*.*

Basic examples of lbo are the empty order $\varnothing$, the one-point order $*$, and the two-point "Sierpinski" order $\Sigma = (\{\top, \bot\}, \sqsubseteq, \neg)$ with $\top \sqsubseteq \bot$ and $\neg\bot = \top, \neg\top = \bot$.

There is a dual to $\uparrow$ — $x \downarrow y$ if $\neg x \uparrow \neg y$, or $x_\bot \sqsubseteq y$ and $y_\bot \sqsubseteq x$. By involutivity of negation, if $x \downarrow y$ then x and y have a least upper bound $\neg(\neg x \wedge \neg y)$ and a greatest lower bound$\neg(\neg x \vee \neg y)$.

The relation $\uparrow$ corresponds to stable coherence or boundedness, and is used to define a stable order.

Definition 2.2 *The* stable *order,* $\leq_s$ *on a locally boolean order is defined:* $x \leq_s y$ *if* $x \uparrow y$ *and* $x \sqsubseteq y$*.*

This is a well-defined partial order; we have the following equivalent characterization of $\leq_s$.

Lemma 2.3 *In a locally boolean order, $x \leq_s y$ if and only if $x \sqsubseteq y \sqsubseteq x^\top$ if and only if $x \sqsubseteq y$ and $y^\top \sqsubseteq x^\top$.*

PROOF: If $x \sqsubseteq y$ and $x \uparrow y$ then $x \sqsubseteq y \sqsubseteq x^\top$ by definition. If $x \sqsubseteq y \sqsubseteq x^\top$ then $\neg y \sqsubseteq \neg x$, and so $y^\top \sqsubseteq x^\top \vee \neg x = x^\top$. If $x \sqsubseteq y$ and $y^\top \sqsubseteq x^\top$, then $x \sqsubseteq y \sqsubseteq y^\top$ and $y \sqsubseteq y^\top \sqsubseteq x^\top$, and so $x \uparrow y$. □

Note that $x \uparrow y$ if and only if x and y are bounded above (by $x \vee y$) in the stable order.

Definition 2.4 *An element p of a lbo is* prime *if $x \uparrow y$, $p \sqsubseteq x \vee y$ implies $p \sqsubseteq x$ or $p \sqsubseteq y$, and $x \wedge y \sqsubseteq \neg p$ implies $x \sqsubseteq \neg p$ or $y \sqsubseteq \neg p$. An element is* finite *if it dominates finitely many elements in the stable order.*

For any element x, we denote the set of prime elements stably less than x by $P(x)$ and the set of finite primes stably less than x by $CP(x)$.

Definition 2.5 *A locally boolean domain is a complete locally boolean order which is prime algebraic — every element x is the (stable) least upper bound of its set of its finite prime approximants $P(x)$.*

From prime algebraicity, we obtain the following characterisation of the extensional order via the primes.

Lemma 2.6 *$x \sqsubseteq y$ iff for all $p \in CP(x)$ there exists $q \in CP(y)$ such that $p \sqsubseteq q$.*

Recall that a dI-domain is a bounded complete and distributive algebraic cpo such that each compact element dominates finitely many elements.

Lemma 2.7 *If A is a locally boolean domain, then $(|A|, \leq_s)$ is a dI-domain.*

PROOF: First note that if X is any set stably bounded above by z, and the $\sqsubseteq$-lub $\bigsqcup X$ exists, then it is also a stable upper bound — for any $x \in X$ we have $y \sqsubseteq x^\top$ for all $y \in X$ and hence $x \leq_s \bigsqcup X$ — and moreover a stable least upper bound. If z is any $\leq_s$ upper bound for X then $\bigsqcup X \sqsubseteq z$, and for any prime $p \leq_s z$, for all $x \in X$ we have $p \sqsubseteq z \sqsubseteq x^\top$, and hence by primeness, $p \sqsubseteq x$ or $p \sqsubseteq \neg x$. So either $p \sqsubseteq x$ for some $x \in X$ — and hence $p \sqsubseteq (\bigsqcup X)^\top$ — or else $p \sqsubseteq \neg x$ for all $x \in X$ — and hence $p \sqsubseteq \neg(\bigsqcup X) \sqsubseteq (\bigsqcup X)^\top$.

Thus if $x, y \leq_s z$, then $x \vee y \leq_s z$, and so every stably bounded finite set has a stable least upper bound, and for any stably bounded X, we may form a stable least upper bound as the $\sqsubseteq$-lub of the directed set of lubs of finite subsets of X. A is algebraic since every finite prime must be compact. Every stably compact element must dominate finitely many elements since it is the least upper bound of the directed set of sups of finite sets of its approximants.

We note also that if $x \uparrow y$, then $x \wedge y$ is the stable least upper bound of x and y, since $x \wedge y \sqsubseteq x, y \sqsubseteq (x \wedge y)^{\top}$ — if $p \leq_s x \sqsubseteq y^{\top}$, then either $p \sqsubseteq y$ or $p \sqsubseteq \neg y$ and so so $p \sqsubseteq (x \wedge y)^{\top}$ — and if $z \leq_s x, y$ then $z \sqsubseteq x \wedge y \sqsubseteq z^{\top}$. Thus to show distributivity, e.g. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, suppose $p \leq_s (x \vee y) \wedge (x \vee z)$. Then $p \leq_s x \vee y$ and hence either $p \leq_s x$ — in which case $p \leq_s x \vee (y \wedge z)$ — or $p \leq_s y$ and $p \leq_s z$ and hence $p \leq_s x \vee (y \wedge z)$. $\Box$

We also observe that every locally boolean domain is a bistable biorder: we define the bistable coherence relation $x \updownarrow y$ if $x^{\top} = y^{\top}$.

Proposition 2.8 *If A is a locally boolean domain, then $(|A|, \sqsubseteq, \updownarrow)$ is a bistable biorder.*

Proof: $\updownarrow$ is clearly an equivalence relation. For any $x, y \in A$, we have $x \updownarrow y$ if and only if $x \uparrow y$ and $x \downarrow y$, and hence $x \updownarrow x \wedge y$ and $x \updownarrow x \vee y$. So $[x]$ is a distributive lattice, inclusion of which into A preserves meets and joins. $\Box$

3 Prenex Normal Forms

The standard product and co-product constructions for partial orders can be extended to locally boolean orders in a natural way.

Definition 3.1 *The product and co-product of an indexed family of lbos $\{A_i \mid i \in I\}$ are lbos defined pointwise — i.e. $\Pi_{i \in I} A_i = (\Pi_{i \in I}|A_i|, \sqsubseteq, \neg)$ (where $\langle x_i \mid i \in I\rangle \sqsubseteq \langle y_i \mid i \in I\rangle$ if $x_i \sqsubseteq y_i$ for all i, and $\neg\langle x_i \mid i \in I\rangle = \langle \neg x_i | \ i \in I\rangle$).*
$\coprod_{i \in I} A_i = (\coprod_{i \in I}|A_i|, \sqsubseteq, \neg)$ (where $\mathtt{in}_i(x) \sqsubseteq \mathtt{in}_j(y)$ if $i = j$ and $x \sqsubseteq y$, and $\neg(\mathtt{in}_i(x)) = \mathtt{in}_i(\neg(x))$).

The other key operation on lbos is *(bi)lifting*, which adds *two* complementary elements, $\top$ and $\bot$.

Definition 3.2 *If A is a locally boolean order, then its lift $A_{\bot}^{\top}$ is the pointed lbo with $|A_{\bot}^{\top}| = \{\mathtt{in}(x) \mid x \in |A|\} \cup \{\bot, \top\}$; $x \sqsubseteq y$ iff $x = \bot$ or $y = \top$ or $x = \mathtt{in}(x')$ and $y = \mathtt{in}(y')$ and $x' \sqsubseteq_A y'$; $\neg\mathtt{in}(x) = \mathtt{in}(\neg(x))$ and $\neg\top = \bot, \neg\bot = \top$.*
The separated sum of an indexed family of locally boolean domains is defined $\Sigma_{i \in I} A_i = (\coprod_{i \in I} A_i)_{\bot}^{\top}$.

Lemma 3.3 *The bilifting, product or coproduct of locally boolean domains is a locally boolean domain.*

Proof: Writing $P(A)$ for the set of primes in A, we have:

$P(A_\bot^\top) = \mathtt{in}(P(A)) \cup \{\bot, \top\}$
$P(\Pi_{i\in I} A_i) = \{x \in \Pi_{i\in I} A_i \mid \exists! i \in I.\pi_i(x) \in P(A_i) \wedge \forall j \neq i.\pi_j(x) = \bot\}$,
$P(\coprod_{i\in I} A_i) = \coprod_{i\in I} P(A_i)$. $\square$

Moreover, every locally boolean domain can be viewed as the limit of a series of approximants constructed using only products, coproducts and lifting, as we shall now show. First, we decompose an arbitrary locally boolean domain into a coproduct of *pointed* locally boolean domains.

Lemma 3.4 *For any element a of a locally boolean domain A there exists a least element $\bot(a)$ such that $\bot(a) \leq_s a$.*

PROOF: If a is compact, then we may take the infimum of the (finite) set of elements stably dominated by a. If a is not compact, we observe that for any two compact aproximants b, c, $\bot(b) \wedge \bot(c) = \bot(b) = \bot(c)$, and therefore $\bot(a) = \bot(b) = \bot(c)$. $\square$

Let $\mathsf{s}(A) = \{\bot(a) \mid a \in A\}$, and for each $x \in A$, let $A_x = \{a \in A \mid \bot(a) = x\}$.

Proposition 3.5 $A \cong \coprod_{x\in \mathsf{s}(A)} A_x$.

PROOF: To show that each A_x is a locally boolean domain, and that the map which sends $a \in A$ to $\mathtt{in}_{\bot(a)}(a)$ is an order-isomorphism, it is sufficient to show that $\bot(a) = \bot(\neg a)$, and that $a \sqsubseteq b$ implies $\bot(a) = \bot(b)$.

For the former, since $(\bot(a))_\bot \leq_s \bot(a)$, minimality implies that $(\bot(a)) = (\bot(a))_\bot$. Hence $(\bot(a))_\bot = \bot(a) \sqsubseteq \neg a \sqsubseteq \bot(a)^\top$ and $\bot(a) \leq_s \neg a$ (and is the least such element) as required.

For the latter, suppose $a \sqsubseteq b$. Then $\bot(a) \uparrow \bot(b)$, since $\bot(a) \sqsubseteq a \sqsubseteq b \sqsubseteq \neg\bot(b) \sqsubseteq \bot(b)^\top$ and $\bot(b) \sqsubseteq \neg b \sqsubseteq \neg a \sqsubseteq \bot(a)^\top$. Hence $\bot(a) \wedge \bot(b) \leq_s a, b$, and so $\bot(a) \wedge \bot(b) = \bot(a) = \bot(b)$. $\square$

Definition 3.6 *A pointed locally boolean domain is said to be* lifted *if its top element is prime and distinct from $\bot$.*

The following lemma justifies the terminology.

Lemma 3.7 *If A is lifted then there is a lbd, $A^\downarrow$ such that $(A^\downarrow)_\bot^\top$.*

PROOF: $A^\downarrow$ is defined in the obvious way, by removing the bottom and top elements of A, so it suffices to show that $A^\downarrow$ is a locally boolean domain. If $x \in |A| - \{\bot, \top\}$, then $\neg x, x^\top \in |A| - \{\bot, \top\}$, since if $x^\top = x \vee \neg x = \top$, then $x = \top$ or $\neg x = \top$ by primality of $\top$, and so $x \in \{\top, \bot\}$. And if $x, y \in A^\downarrow$ and $x \uparrow y$ or $x \downarrow y$, then $x \vee y \in A^\downarrow$, since if $x \vee y = \bot$ then $x = y = \bot$ and if $x \vee y = \top$, then $x = \top$ or $y = \top$ by primality of $\top$. Similarly, $x \wedge y = \neg(\neg x \vee \neg y) \in A^\downarrow$. $\square$

Finally we can show that every pointed locally boolean domain can be decomposed as a product of lifted domains. As an indexing set, we take $\mathsf{g}(A)$, the set of $\sqsubseteq$-maximal prime approximants to $\top$.

Lemma 3.8 *If $p, q \in P(\top)$ and $p \sqsubseteq q$, then either $p = q$ or $p = \bot$.*

PROOF: We have $q \sqsubseteq p^\top = \top$, and hence $q \sqsubseteq p$ or $q \sqsubseteq \neg p$, in which case $p \sqsubseteq \neg q$, and $p = p \wedge q \sqsubseteq \neg q \wedge q = \bot$. $\square$

Hence $\mathsf{g}(A)$ is non-empty. We note that for any $p \in \mathsf{g}(A)$, $p \sqsubseteq \top \sqsubseteq p^\top$ and so $p^\top = \top$ and $p_\bot = \bot$.

Definition 3.9 *For each $p \in \mathsf{g}(A)$, we define a partial order with involution $(A_p, \sqsubseteq_p, \neg_p)$ by letting $|A_p| = \{x \in |A| \mid x \sqsubseteq p\}$, $\sqsubseteq_p = \sqsubseteq\restriction A_p$ and $\neg_p(x) = p \wedge \neg x$, which is well-defined since $\neg x \downarrow p$ (as $p_\bot = \bot \sqsubseteq \neg x$ and $(\neg x)_\bot = x_\bot \sqsubseteq x \sqsubseteq p$).*

Lemma 3.10 *For each p, A_p is a locally boolean order.*

PROOF: $\neg_p$ is clearly antitone. It is also involutive: for any $x \sqsubseteq p$, $\neg_p(\neg_p(x)) = p \wedge \neg(p \wedge \neg x) = p \wedge ((\neg p) \vee x) \geq x$. To show $\neg_p \neg_p x \sqsubseteq x$, suppose $q \leq_s \neg_p \neg_p x$, then $q \sqsubseteq p$, and $q \sqsubseteq \neg p \vee x$, and by primeness, either $q \sqsubseteq x$ or else $q \sqsubseteq \neg p$, but in the latter case $q \sqsubseteq p_\bot = \bot$, and so $q \sqsubseteq x$ as required.

For each $x \in A_p$, we have $\downarrow \{x, \neg x, p\}$, and $x \vee \neg_p x = x \vee (\neg(x) \wedge p) = x^\top \wedge p$, which is in A_p.

If we have $x \uparrow y$ in A_p — i.e. $x \sqsubseteq y^\top \wedge p$ and $y \sqsubseteq x^\top \wedge p$ then $x \uparrow y$ in A, and $x \wedge y$ and $x \vee y$ are both in A_p. $\square$

Lemma 3.11 *$q \in A_p$ is prime in A if and only if it is prime in A_p.*

PROOF: If q is prime in A, then if $a \uparrow b$ (or $a \downarrow b$) in A_p, then $a \uparrow b$ (or $a \downarrow b$) in A and so if $q \sqsubseteq a \vee b$, $q \sqsubseteq a$ or $q \sqsubseteq b$ as required.

Suppose q is prime in A_p. We have $q = \bigvee X$, where $X = \{r \in P(A) \mid r \leq_s q\}$. Since $r \leq_s q$ implies $r \in A_p$, and $r \leq_s q$ in A implies $r \leq_s q$ in A_p, X is a stably bounded subset of A_p. Hence by primeness of p in A_p, $p = r$ for some $r \in X$ and hence p is prime in A. $\square$

So A_p is a locally boolean domain. Note also that if A has at least two elements, then for each $p \in \mathsf{g}(A)$, A_p is lifted, since p is prime in A_p, and distinct from $\bot$ because $\bot \notin \mathsf{g}(A)$.

Lemma 3.12 *For all primes $r \in A$ there exists $p \in \mathsf{g}(A)$ such that $r \sqsubseteq p$. Moreover, if $r \neq \bot$ then p is unique.*

Proof: We have $r \sqsubseteq \top$ and hence there exists $q \leq_s \top$ such that $r \sqsubseteq q$, and $p \in \mathtt{g}(A)$ such that $q \leq_s p$, and hence $r \sqsubseteq p$.

To show uniqueness of p, we observe that for distinct $p, q \in \mathtt{g}(A)$, $p \uparrow q$ (as $p^\top = q^\top = \top$) and $p \wedge q = \bot$, since $p \sqsubseteq q \vee \neg q = \top$, so by primality and maximality of p, $p \sqsubseteq \neg q$, hence $p \wedge q \sqsubseteq \neg q \wedge q = q_\bot = \bot$. Thus if $r \sqsubseteq p, q$ then $r = \bot$. □

Proposition 3.13 $A \cong \Pi_{p \in \mathtt{g}(A)} A_p$.

Proof: We define a map $\phi : A \to \Pi_{p \in \mathtt{g}(A)} A_p$ by $\phi(x) = \langle \bigvee \{q \in A_p \mid q \leq_s x\} \mid p \in \mathtt{g}(A)\rangle$. To show that this is an order-isomorphism, we first establish that it restricts to a (stable-boundedness preserving) order-isomorphism on primes.

- If r is prime then $\phi(r)$ is prime — if $r \neq \bot$ then by Lemma 3.12 there exists unique $p \in \mathtt{g}(A)$ such that $r \sqsubseteq p$, and so $\pi_p(\phi(r)) = r$ (which is prime in A_p), and $\pi_q(\phi(r)) = \bot$ for $q \neq p$, i.e. $\phi(r)$ is prime.
- If $r \sqsubseteq q$, then $\phi(r) \sqsubseteq \phi(q)$ since either $r = \bot$ or there exists unique p, p' such that $r \sqsubseteq p$ and $q \sqsubseteq p'$ and hence $p = p'$ and so $\phi(r) \sqsubseteq \phi(q)$ as required,
- For any prime $q \in \Pi_{p \in \mathtt{g}(A)} A_p$ there exists a *unique* prime $\phi^{-1}(q) \in A$ such that $\phi(\phi^{-1}(q)) = q$, since if $q \neq \bot$, there exists a unique $p \in \mathtt{g}(A)$ such that $\pi_p(q) = r$ is a non-$\bot$ prime of A_p, and hence by Lemma 3.11, r is a prime of A such that $\phi(r) = p$.
- For any primes r', r', $\phi(r) \uparrow \phi(r)$ if and only if $r \uparrow r'$ — suppose $\phi(r) \uparrow \phi(r')$, then either $r, r' \in A_p$ for some $p \in \mathtt{g}(A)$, and $r \uparrow r'$ in A_p and hence in A by Lemma 3.11, or else $r \in A_p$ and $r' \in A_q$ where $p \neq q$. In the latter case, by Lemma 3.12, $r \not\sqsubseteq q$ and $r' \not\sqsubseteq p$. So $r' \sqsubseteq \top = p^\top$ implies $r' \sqsubseteq \neg p$ by primeness of r, and hence $r' \sqsubseteq \neg r \sqsubseteq r^\top$ and vice-versa. For the converse, we have already remarked in the proof of Lemma 3.10 that stable boundedness in A implies stable boundedness in A_p.

Since ϕ is determined by its action on primes — i.e. $\phi(x) = \bigvee \{\phi(p) \mid p \leq_s x\}$ — it is an order-isomorphism: we may define $\phi^{-1}(x) = \bigvee \{\phi^{-1}(p) \mid p \leq_s x\}$. □

By putting together Propositions 3.5, 3.7 and 3.13 we obtain the following.

Corollary 3.14 *If A is pointed, then* $A \cong \Pi_{p \in \mathtt{g}(A)} \Sigma_{x \in \mathtt{s}(A_p^\downarrow)} A_{px}$.

We will say that a locally boolean domain A is *bounded* if there is a (least) natural number n (the *rank* of A) such that every prime in A stably dominates at most n elements. By repeatedly applying Corollary 3.14, we can show that every bounded (and pointed) locally boolean domain is isomorphic to a *prenex normal form* in which Π and Σ alternate. Formally, we define the set of $\Pi\Sigma$-orders to be the smallest set such that for any family of families (possibly empty) of $\Pi\Sigma$-orders, $\{\{A_{ij} \mid j \in J_I\} \mid i \in I\}$, $\Pi_{i \in I} \Sigma_{j \in J} A_{ij}$ is a $\Pi\Sigma$-order.

Given any pointed lbd A, we use Corollary 3.14 to define a series $\{A^i \mid i \in \omega\}$ of $\Pi\Sigma$-orders, as follows: $A^0 = *$ and $A^{i+1} = \Pi_{p\in \mathtt{g}(A)}\Sigma_{x\in \mathtt{s}(A^{\downarrow}_p)}A^i_{px}$. We prove the following by induction on rank.

Proposition 3.15 *If A has rank n, then $A \cong A^n$.*

To extend our results to unbounded locally boolean domains, we show that every such domain is the limit of the chain of $(\Pi\Sigma)$ cpos $\{A^i \mid i \in A_i\}$ in a standard sense. Recall that an embedding/projection (e-p) pair between cpos A and B is a pair of $\sqsubseteq$-continuous functions $\iota : A \to B$ and $\pi : B \to A$ such that $\iota;\pi = \mathtt{id}_A$ and $\pi;\iota \sqsubseteq \mathtt{id}_B$. A limit (or co-limit) for an ω-chain of cpos — a family $\{A_i \mid i \in \omega\}$ with e-p pairs $\iota_i : A_i \trianglelefteq A_{i+1} : \pi_i$ for each i — is a cpo B with e-p pairs $\iota^i : A_i \trianglelefteq B : \pi^i$ for each i such that $\pi^{i+1};\pi_i = \pi^i$, and for every $b \in B$, we have $b = \bigsqcup\{\iota^i(\pi^i(b)) \mid i \in \omega\}$. Clearly, B is unique up to isomorphism.

Lemma 3.16 *For every $i \in \omega$ there is a e-p pair $\iota^i_A : A^i \trianglelefteq A : \pi^i_A$.*

Proof: We define the embedding and projection inductively, using Corollary 3.14 — for any A, $\iota^0_A = \bot$ and $\pi^0_A = \bot$, whilst $\iota^{i+1}_A = \phi_A; (\Pi_{p\in\mathtt{g}(A)}\Sigma_{x\in\mathtt{s}(A^{\downarrow}_p)}\iota^i_{A_{px}})$ and $\pi^{i+1}_A = (\Pi_{p\in\mathtt{g}(A)}\Sigma_{x\in\mathtt{s}(A^{\downarrow}_p)}\pi^i_{A_{px}}); \phi^{-1}_A$. □

Lemma 3.17 *If $q \in P(A)$ (stably) dominates at most n elements then $q = \iota^n(\pi^n_A(q))$.*

Proof: is by induction on n. If q dominates at most $n+1$ elements, then so does $\phi(q)$, which is either $\bot$ or $\top$, or else has pth component $\mathtt{in}_x(r)$ for some $p \in \mathtt{g}(A)$ and $x \in \mathtt{s}(A^{\downarrow}_p)$. r dominates at most n elements, and so by induction, $r = \iota^n_{A_{px}}(\pi^n_{A_{px}}(r))$. Hence $q = \iota^{n+1}(\pi^{n+1}_A(q))$ as required. □

Theorem 3.18 *[Decomposition Theorem]Any pointed lbd A is a limit for the ω-chain $\{A^i \mid i \in \omega\}$.*

Proof: For each i, we have e-p pairs $\iota^i : A^i \trianglelefteq A : \pi^i$ such that $\pi^{i+1};\pi_i = \pi^i$. By Lemma 3.17, for every finite prime of A, there exists $n \in \omega$ and $q \in A^n$ such that $p = \iota^i(q)$. Hence for every $a \in B$, we have $a = \bigsqcup\{\iota^i(\pi^i(a)) \mid i \in \omega\}$ as required. □

Corollary 3.19 *Every locally boolean domain is the limit of an ω-chain in which every element is a co-product of $\Pi\Sigma$-orders.*

We will also show that for certain "locally boolean" ω-chains of lbds, we can define an involution operator on the limit, making it a locally boolean domain (we use this fact to establish that various function-spaces of lbds are locally boolean domains),

Definition 3.20 *A homomorphism of locally boolean orders is a continuous function which preserves all locally boolean structure — i.e. if* $f(\neg x) = \neg f(x)$, $f(x^\top) = f(x)^\top$, *and if* $x \uparrow y$, *then* $f(x \vee y) = f(x) \vee f(y)$ *and* $f(x \wedge y) = f(x) \wedge f(y)$.
We shall say that a ω*-chain* $\{A_i \mid i \in \omega\}$ *is* locally boolean *if for all* i, $\pi_i : A_{i+1} \rightarrow A_i$ *is a locally boolean homomorphism, and for all* x, $\iota_i(\pi_i(x)) \leq_s x$.

Proposition 3.21 *If* A *is a limit for a locally boolean* ω*-chain, then* A *is (isomorphic to) a locally boolean domain.*

PROOF: For any x, the set $X = \bigsqcup\{\iota^i(\neg\pi^i(x)) \mid i \in \omega\}$ is directed, since $\iota^i(\neg\pi^i(x)) = \iota^i(\neg(\pi_i(\pi^{i+1}(x))))) = \iota^i(\pi_i(\neg(\pi^{i+1}(x)))) \sqsubseteq \iota^{i+1}(\neg(\pi^{i+1}(x)))$. Thus we may define $\neg x = \bigsqcup X$. This operation is clearly antitone, and by continuity of π, for any i we have $\pi^i(\neg x) = \neg\pi^i(x)$. Thus for all i, $\pi^i(\neg\neg x) = \neg\neg\pi^i(x) = x$, and hence $\neg\neg x = x$.

For any x, $\bigsqcup\{\iota^i(\pi^i(x)^\top) \mid i \in \omega\}$ is well-defined and is a least upper bound for $x, \neg x$, since for every i, $\pi^i(\bigsqcup\{\iota^i((\pi^i(x))^\top) \mid i \in \omega\}) = \pi^i(x)^\top$. If $x \uparrow y$, then $x \vee y$ and $x \wedge y$ are given by $\bigsqcup\{\iota^i(\pi^i(x) \vee \pi^i(y)) \mid i \in \omega\}$ and $\bigsqcup\{\iota^i(\pi^i(x) \wedge \pi^i(y)) \mid i \in \omega\}$.

For each prime $p \in A^i$, $\iota^i(p)$ is prime in A, since if $x \uparrow y$ or $x \downarrow y$, and $\iota^i(p) \sqsubseteq x \vee y$ then $p = \pi^i(\iota^i(p)) \sqsubseteq \pi^i(x) \vee \pi^i(y)$, and either $p \sqsubseteq \pi(x)$ — so $\iota^i(p) \sqsubseteq \iota^i(\pi^i(x)) \sqsubseteq x$ — or $p \sqsubseteq \pi^i(y)$ — and $\iota^i(p) \sqsubseteq y$.

Note that if $y \leq_s x$ in A^i, then $\iota_i(y) \leq_s \iota_i(x)$ in A^{i+1} as $\iota_i(y) \sqsubseteq \iota_i(x) \sqsubseteq \iota_i(y^\top) = \iota_i(\pi_i(\iota_i(y))^\top) = \iota_i(\pi_i(\iota_i(y)^\top)) \sqsubseteq \iota_i(y)^\top$. Hence if $p \leq_s \pi^i(y)$, then for each $j \geq i$, $\iota^i_j(p) \leq_s \iota^i_j(\pi^i(y)) = \iota^i_j(\pi^i_j(\pi^j(y))) \leq_s \pi^j(y)$, and hence $\iota^i(p) \sqsubseteq y \sqsubseteq (\iota^i(p))^\top$ — i.e. $\iota^i(p) \leq_s y$. Thus each $x \in A$ is the least upper bound of the set of finite primes $\{\iota^i(p) \mid p \in CP(\pi^i(y)), i \in \omega\}$. $\square$

We note that the terminal morphism from A to $*$ is a locally boolean homomorphism for any A, and that for any family $\{f : A_i \rightarrow B_i \mid i \in I\}$ of locally boolean homomorphisms, $\Pi_{i \in I} f_i : \Pi_{i \in I} A_i \rightarrow \Pi_{i \in I} B_i$ and $\Sigma_{i \in I} f_i : \Sigma_{i \in I} A_i \rightarrow \Sigma_{i \in I} B_i$ are locally boolean homomorphisms. So, for example, for each A, the chain of $\Pi\Sigma$ approximants $\{A^i \mid i \in \omega\}$ is locally boolean.

4 A Model of Linear Logic

We shall now define a semantics of linear logic based on locally boolean domains, and affine and linear functions. Whilst not capturing full classical linear logic (this is precluded by its sequential nature) our model does have an interesting duality property (reflecting the Player/Opponent duality which features in games models).

Recall that a function f is bistable if $x \updownarrow y$ (i.e. $x^\top = y^\top$) implies $f(x) \updownarrow f(y)$, $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$. We define notions of bistable function which are (respectively) affine and linear with respect to the stable coherence induced on locally boolean domains.

Definition 4.1 *A $\sqsubseteq$-continuous and bistable function f is* affine *if $a \uparrow b$ implies $f(a \wedge b) = f(a) \wedge f(b)$ (stability) and $f(a \vee b) = f(a) \vee f(b)$.* [1]

If A, B are pointed, then $f : A \to B$ is strict *if $f(\top) = \top$ and $f(\bot) = \bot$; f is* linear *if it is affine and strict (i.e. it preserves glbs and lubs for all bounded sets).*

Thus we have the following categories of locally boolean domains.

- $\mathcal{LBB}$, the category of locally boolean domains and bistable continuous functions,
- $\mathcal{LBA}$, the category of locally boolean domains and affine functions,
- $\mathcal{PLBA}$, the category of pointed locally boolean domains and affine functions,
- $\mathcal{PLBL}$, the category of pointed locally boolean domains and linear functions,

Clearly, $\mathcal{PLBL}$ is a subcategory of $\mathcal{PLBA}$, which is a full subcategory of $\mathcal{LBA}$, which is a subcategory of $\mathcal{LBB}$. As in standard accounts of domain theory, lifting is left adjoint to the inclusion of $\mathcal{LLBA}_\mathcal{S}$ in $\mathcal{LBA}$. We write $\mathtt{up} : I \to (\_)^\top_\bot$ for the unit of this adjunction.

Lemma 4.2 *$\mathcal{PLBL}$ is a reflective subcategory of $\mathcal{LBA}$ — for pointed B, there is a natural isomorphism between $\mathcal{PLBL}(A^\top_\bot, B)$ and $\mathcal{LBA}(A, B)$.*

The indexed product and co-product constructions also have the expected properties.

Proposition 4.3 *$\mathcal{LBA}$ and $\mathcal{PLBL}$ have all (small) products and (distributive) coproducts, and $\mathcal{PLBA}$ has all products.*

We now define symmetric monoidal closed structure on these categories.

Definition 4.4 *For any locally boolean domains, A and B, we define the cpo $A \multimap B$ to be the set of affine functions from A to B, ordered extensionally. For any* pointed *lbds A and B the cpo $A \multimap B$ is the set of linear functions from $A \multimap B$.*

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[1] Note that if f is bistable, then $a \uparrow b$ already implies $f(a) \uparrow f(b)$, since $f(a) \sqsubseteq f(b^\top) \sqsubseteq f(b)^\top$ and vice-versa.

It is straightforward to show that $\multimap$ and $\rightarrowtriangle$ are (mixed variance) functors into the category of cpos and continuous functions. To establish that the affine and linear function spaces are locally boolean domains, we need to define an appropriate involution on functions. The obvious choice is to set $(\neg f)(x) = \neg f(\neg x)$. In fact, this is in general a continuous function only if A and B are bounded lbds[2] — reflecting the asymmetry in the computational behaviour of $\bot$ (non-termination) and $\top$ (immediate termination). Thus we need to define $\neg f$ via its action on compact elements — $(\neg f)(x) = \bigsqcup\{\neg f(\neg p) \mid p \in CP(x)\}$. To prove that this is well-defined, and that $A \multimap B$ and $A \rightarrowtriangle B$ are locally boolean domains we will show that they are locally boolean limits of chains of $\Pi\Sigma$-orders. In doing so, we axiomatize the intensional structure of the model — and hence the correspondence with more intensional models such as categories of games and sequential algorithms/strategies — via a series of natural isomorphisms connecting $\multimap$ and $\rightarrowtriangle$ with the the lifting, product and co-product operations.

Lemma 4.5 *For any B, and pointed A, there is a natural isomorphism $A \multimap B_\bot^\top \cong (A \rightarrowtriangle B_\bot^\top + A \multimap B)_\bot^\top$.*

PROOF: We have a natural map from $(A \rightarrowtriangle B_\bot^\top + A \multimap B)_\bot^\top$ to $A \multimap B_\bot^\top$ which sends $\mathtt{in_l}(f)$ to f, and $\mathtt{in_r}(f)$ to $f;\mathtt{up}_B$. To establish that this is an isomorphism we first show that if $f : A \to B_\bot^\top$ is non-strict, non-$\bot$ and non-$\top$, then $f = \widehat{f};\mathtt{up}$ for some $\widehat{f} : A \to B$. This follows from the observation that for all $x \in A$, $f(x) \notin \{\bot, \top\}$, since if e.g. $f(x) = \top$, then $f(\top) = \top$, and hence also $f(\bot) \in \{\bot, \top\}$, as $f(\top) \updownarrow f(\bot)$. But if $\bot = \bot$, then f is strict, and if $f(\bot) = \top$ then f is constantly $\top$, both of which contradict our assumptions.

If f is strict and g is non-strict, non-$\top$ and non-$\bot$, then $g \not\sqsubseteq f$, and $f \not\sqsubseteq g$. Thus the strict map which sends each strict function f to $\mathtt{in_l}(f)$, and each non-strict, non-$\bot$ and non-$\top$ function to $\mathtt{in_r}(\widehat{f})$, is the required inverse. □

Lemma 4.6 *If A is pointed, then for any family $\{B_i \mid i \in I\}$, there is a natural isomorphism $A \multimap \coprod_{i \in I} B_i \cong \coprod_{i \in I}(A \multimap B_i)$.*

PROOF: Given $f : A \to \coprod_{i\in I} B_i$, $f(\top) = \mathtt{in}_i(y)$ for some $i \in I$ and $y \in B_i$, and for all $x \in A$, $f(x) \sqsubseteq \mathtt{in}_i(y)$, and so $f(x) = \mathtt{in}_i(z_x)$ for some $z_x \in B_i$. So we can define $\widehat{f} : A \to B_i$ such that $f = \widehat{f};\mathtt{in}_i$ by letting $\widehat{f}(x) = z_x$. So $A \multimap \coprod_{i\in I} B_i \cong \coprod_{i\in I}(A \multimap B_i)$ by the isomorphism which sends f to $\mathtt{in}_i(\widehat{f})$. □

Definition 4.7 *Let $\{A_i \mid i \in I\}$ and B be pointed locally boolean orders. A map $f : \Pi_{i\in I}A_i \to B$ is i-strict if $f(x) = \bot$ whenever $\pi_i(x) = \bot$ and $f(x) = \top$ whenever $\pi_i(x) = \bot$.*

[2] A concrete counterexample has been given by Thomas Streicher.

Lemma 4.8 *If $f : \Pi_{i\in I}A_i \to C_\perp^\top$ is strict and bistable, then f is i-strict for some unique $i \in I$.*

Proof: Given $j \in I$ and $a \in A_j$, define $\perp[a]_j = \Pi_{i\in I}A_i$ by $\pi_j(\perp[a]_j) = a$, and $\pi_i(\perp[a]_j) = \perp$ for $i \neq j$. Let $\top[a]_j = \neg(\perp[\neg a]_j)$.

Then if $\pi_i(x) = \perp$, $x \sqsubseteq \top[\perp]_i$, and if $\pi_i(x) = \top$, $\perp[\top]_i \sqsubseteq x$, and so f is i-strict if $f(\top[\perp]_i) = \perp$, and $f(\perp[\top]_i) = \top$. Since $\perp[\top]_j \updownarrow \perp[\top]_k$ for all j, k, and $\bigvee_{i\in I}(\perp[\top]_i) = \top$, by bistability and continuity, $\bigvee_{i\in I} f(\perp[\top]_i) f(\bigvee_{i\in I}\perp[\top]_i) = \top$, and so $f(\perp[\top]_j) = \top$ for some j. Moreover, $\perp[\top]_j \updownarrow \top[\perp]_j$, and $f(\perp[\top]_j \wedge \top[\perp]_j) = f(\perp) = \perp$, and so $f(\perp[\top]_j) \wedge f(\top[\perp]_j) = \top \wedge f(\top[\perp]_j) = \perp$, and so $f(\top[\perp]_j) = \perp$. □

Lemma 4.9 *If $f : \Pi_{i\in I}A_i \to B_\perp^\top$ is affine and i-strict then $f = \pi_i; g$ for some $g : A_i \to B^top_\perp$.*

Proof: Suppose $f : \Pi_{i\in I}A_i \to B_\perp^\top$ is i-strict. Then for all $a \in A_i$, $\perp[a]_i \uparrow \top[\perp]_i$, and so $f(\top[a]_i) = f(\perp[a]_i \vee \top[\perp]_i) = f(\perp[a]_i) \vee \perp = f(\perp[a]_i)$. For all $x \in \Pi_{i\in I}A_i$, we have $\perp[\pi_i(x)]_i \sqsubseteq x \sqsubseteq \top[\pi_i(x)]_i$ and hence $f(x) = f(\perp[\pi_i(x)]_i)$. Thus $f = \pi_l; g$, where $g(x) = f(\perp[x]_i)$. □

Corollary 4.10 *There is a natural iso $(\Pi_{i\in I}A_i) \twoheadrightarrow B_\perp^\top \cong \coprod_{i\in I}(A_i \twoheadrightarrow B_\perp^\top)$.*

Now we can express a mutual decomposition of $\multimap$ and $\twoheadrightarrow$ into the Π and Σ operations via a pair of dual isomorphisms. (We write $\Sigma_{i\in I}A_i \oplus \Sigma_{j\in J}B_j$ for the sum $\Sigma_{k\in I+J}C_k$, where $C_{inl(i)} = A_i$ and $C_{\mathtt{in_r}(j)} = B_j$.)

Proposition 4.11 *Let $A = \Pi_{i\in I}A_i$, where each A_i is lifted, and let $B = \Pi_{k\in K}\Sigma_{l\in L_k}B_{kl}$, where each B_{kl} is pointed, then there is a natural isomorphism:*

$$A \multimap B \cong \Pi_{k\in K}(\Sigma_{i\in I}(A_i \twoheadrightarrow \Sigma_{l\in L_k}B_{kl}) \oplus \Sigma_{l\in L_k}(\Pi_{i\in I}A_i \multimap B_{kl}))$$

Let $A = \Sigma_{i\in I}\Pi_{j\in J}A_{ij}$, where each A_{ij} is lifted, and let $B = \Sigma_{k\in K}B_k$ where each B_k is pointed. Then there is a natural isomorphism:

$$A \twoheadrightarrow B \cong \Pi_{i\in I}(\Sigma_{j\in J}(A_{ij} \multimap \Sigma_{k\in K}B_k) \oplus \Sigma_{k\in K}(\Sigma_{j\in J}A_{ij} \twoheadrightarrow B_k))$$

Proof: We have $A \multimap B \cong \Pi_{k\in K}(A \multimap \Sigma_{l\in L_k}B_{kl})$, and for each $k \in K$, $A \multimap \Sigma_{l\in L_k}B_{kl} \cong (A \twoheadrightarrow \Sigma_{l\in L_k}B_{kl})_\perp^\top \oplus (A \multimap \coprod_{l\in L_k} B_{kl})_\perp^\top$ by Lemma 4.5.
$A \twoheadrightarrow \Sigma_{l\in L_k}B_{kl} \cong \coprod_{i\in I}(A_i \twoheadrightarrow \Sigma_{l\in L_k}B_{kl})$ by Lemma 4.10,
$(A \multimap \coprod_{l\in L_k} B_{kl} \cong \coprod_{l\in L_k}(A \multimap B_{kl})$, by Lemma 4.6. Hence $A \multimap B \cong \Pi_{k\in K}(\Sigma_{i\in I}(A_i \twoheadrightarrow \Sigma_{l\in L_k}B_{kl}) \oplus \Sigma_{l\in L_k}(\Pi_{i\in I}A_i \multimap B_{kl}))$ as required. The decomposition of $\twoheadrightarrow$ is similar. □

Corollary 4.12 *For any bounded lbds A and B, $A \multimap B$ is order-isomorphic to a prenex normal form and is therefore a locally boolean domain.*

To show that if A and B are unbounded then $A \multimap B$ is a locally boolean domain, we construct a locally boolean chain of approximants and show that $A \multimap B$ is a limit for it.

Definition 4.13 *Given pointed lbds A, B and lifted lbds C, D, we define* locally boolean *ω-chains $\{(A \multimap B)_i \mid i \in \omega\}$ and $\{(C \multimap D)_i \mid i \in \omega\}$ by $(A \multimap B)_0 = (C \multimap D)_0 = *$, and $(A \multimap B)_{i+1} =$
$\Pi_{p\in \mathtt{g}(B)}(\Sigma_{q\in \mathtt{g}(A)}(A_q \multimap \Sigma_{x\in \mathtt{s}(B_p)} B_{px}) \oplus \Sigma_{x\in \mathtt{s}(B_p)}(\Pi_{q\in \mathtt{g}(A)} A_p \multimap B_{px}))$
$(C \multimap D)_{i+1} =$
$\Pi_{x\in \mathtt{s}(C)}(\Sigma_{p\in \mathtt{g}(C_x)}(C_{xp} \multimap \Sigma_{y\in \mathtt{s}(D)} D_y) \oplus \Sigma_{y\in \mathtt{s}(D)}(\Sigma_{p\in \mathtt{g}(C_x)} C_{xp} \multimap D_y))$*

For each i we have e-p pairs (e_i, p_i) between $(A \multimap B)_i$ and $A \multimap B$, for any pointed A, B and between $(C \multimap D)_i$ and $C \multimap D$ for any lifted C, D, defined inductively using Corollary 3.14 and Proposition 4.11.

Lemma 4.14 *For any affine or linear function f, $f = \bigsqcup\{e_i(p_i(f)) \mid i \in \omega\}$.*

Proof: For any compact element x of A we write $|x|$ for the number of elements dominated by x. We show that for any compact elements $a \in A$ and $b \in B$, if $b \leq_s f(a)$ and $i > |a| + |b|$ then $b \leq_s e_i(p_i(f))(a)$, by induction on $|a| + |b|$. The induction follows decomposition of f via Proposition 4.11 — if e.g. $f \in \Sigma_{i\in I}\Pi_{j\in J_i} A_{ij} \multimap \Sigma_{k\in K} B_k$, then if $a = \bot$ then $f(a) = e_i(p_i(f))(a) = \bot$, and similarly, if $a = \top$. So suppose $a = \mathtt{in}_i(a')$, where $|a'| < |a|$. Then $f(a) = (\mathtt{in}_i; f)(a')$, and we may apply the induction hypothesis to $\mathtt{in}_i; f$.

If $f \in \Pi_{i\in I} A_i \multimap \Pi_{j\in J}\Sigma_{k\in K_j} B_k$, then we show that $\pi_j(b) \leq_s \pi_j(e_i(p_i(f))(a))$ for each j: if $f; \pi_j$ is strict, then it is i-strict for some $i \in I$, and equal to $\pi_i; g$ for some g and so the argument above applies to $g : A_i \to \Sigma_{k\in K_j} B_{jk}$ and $\pi_i(a)$. If f is non-strict and non-constant, then $f = g; \mathtt{in}_k$ for some k and if $b \neq \bot, \top$, then $b = \mathtt{in}_k(b')$ with $|b'| < |b|$ hence we can apply the induction hypothesis to a, b' and g. □

Proposition 4.15 *For any pointed lbds A and B, $A \multimap B$ is a limit for $\{(A \multimap B)_i \mid i \in \omega\}$ and for lifted lbds C and D, $C \multimap D$ is a limit for $\{(C \multimap D)_i \mid i \in \omega\}$.*

Thus we may derive an involution operation as described in Proposition 3.21 — $\neg f = \bigsqcup\{e_i(\neg p_i(f)) \mid i \in \omega\}$. This coincides with the direct definition of negation and therefore the latter does satisfy the axioms for a locally boolean domain.

Lemma 4.16 *If x is compact, then $(\neg f)(x) = \neg(f(\neg x))$.*

Proof: is by induction on $|x|$, by following the proof of Proposition 4.11. □

Hence by continuity $(\neg f)(x) = \bigsqcup\{\neg f(\neg p) \mid p \in CP(x)\}$.

Proposition 4.15 extends to arbitrary lbds A, B; using Lemma 4.6 to express them as coproducts of pointed lbds $A \cong \coprod_{i \in I} A_i$ and $B \cong \coprod_{j \in J}$ we have $A \multimap B \cong \coprod_{j \in J} \Pi_{i \in I}(A_i \multimap B_j)$. In fact, we may deduce from this that $\mathcal{LBA}$ is equivalent to the completion of $\mathcal{PLBA}$ with all (small) coproducts — the category $\mathsf{Fam}(\mathcal{PLBA})$ in which objects are set-indexed families of pointed lbds, and morphisms from $\{A_i \mid i \in I\}$ to $\{B_j \mid j \in J\}$ are pairs consisting of a reindexing function $f : I \to J$ and a family of morphisms $\{\phi_i : A_i \to B_{f(i)} \mid i \in I\}$ (see [1] for a general account of this construction applied to game semantics).

By considering Proposition 4.11 in the special case in which $B = \Sigma$, we may observe the following.

Proposition 4.17 *For any lbd A, there is a natural isomorphism $A \cong (A \multimap \Sigma) \twoheadrightarrow \Sigma$ and if A is pointed there is a natural isomorphism $A \cong (A \twoheadrightarrow \Sigma) \multimap \Sigma$.*

Proof: We prove this for $\Pi\Sigma$ orders by structural induction using Proposition 4.11:
For example $((\Pi_{i \in I}\Sigma_{j \in J_i} A_{ij}) \multimap \Sigma) \twoheadrightarrow \Sigma \cong (\Sigma_{i \in I}(\Sigma_{j \in J_i}(A_{ij} \twoheadrightarrow \Sigma)) \twoheadrightarrow \Sigma \cong \Pi_{i \in I}((\Sigma_{j \in J_i}(A_{ij}) \twoheadrightarrow \Sigma) \multimap \Sigma) \cong \Pi_{i \in I}(\Pi_{j \in J_i}(A_{ij} \multimap \Sigma) \multimap \Sigma) \cong \Pi_{i \in I}\Sigma_{j \in J_i}((A_{ij} \multimap \Sigma)) \twoheadrightarrow \Sigma) \cong \Pi_{i \in I}\Sigma_{j \in J_i} A_{ij}$.
This generalizes to arbitrary lbds by continuity and the Decomposition Theorem. $\Box$

In other words, the adjunction between the functors $\_ \multimap : \mathcal{LBA} \to \mathcal{PLBL}^{OP}$ and $\_ \twoheadrightarrow : \mathcal{PLBL}^{OP} \to \mathcal{LBA}$ is an equivalence.

Corollary 4.18 *The categories $\mathcal{LBA}$ and $\mathcal{PLBL}$ are dual.*

We can now use duality to derive the symmetric monoidal structure on $\mathcal{LBA}$ from $\multimap, \twoheadrightarrow$ and Σ.

Definition 4.19 *We define a bifunctor $\otimes$ on $\mathcal{LBA}$ by $\_ \otimes \_ = (\_ \multimap (\_ \multimap \Sigma)) \twoheadrightarrow \Sigma$.*

The unit of the tensor, I, is the one-point domain (which is terminal, so the model is affine). We have $A \otimes I = (A \multimap (I \multimap \Sigma)) \twoheadrightarrow \Sigma \cong (A \multimap \Sigma) \twoheadrightarrow \Sigma \cong A \cong I \otimes A$, $A \otimes B = (A \multimap B \multimap \Sigma) \twoheadrightarrow \Sigma \cong (B \multimap A \multimap \Sigma) \twoheadrightarrow \Sigma = B \otimes A$, and $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$.

Proposition 4.20 $(\mathcal{LBA}, I, \otimes)$ *is a SMCC.*

Proof: We have a natural isomorphism: $\mathcal{LBA}(A \otimes B, C) = \mathcal{LBA}((A \multimap (B \multimap \Sigma)) \twoheadrightarrow \Sigma, C) \cong \mathcal{PLBL}(C \multimap \Sigma, A \multimap (B \multimap \Sigma)) \cong \mathcal{LBA}(A, (C \multimap \Sigma) \twoheadrightarrow (B \multimap \Sigma)) \cong \mathcal{LBA}(A, B \multimap (C \multimap \Sigma) \twoheadrightarrow \Sigma) \cong \mathcal{LBA}(A, B \multimap C)$. $\Box$

The dual $\mathbin{⅋}$ to $\otimes$, which is a symmetric premonoidal functor on $\mathcal{PLBA}$ (and of course a bifunctor on $\mathcal{PLBL}$), is defined on pointed objects: $A⅋B = (A \rightarrow \Sigma) \multimap B$, and thus $A⅋B \cong ((A \rightarrow \Sigma) \otimes (B \rightarrow \Sigma)) \multimap \Sigma$. We thus have a model for polarized MALL [13].

Proposition 4.11 allows us to fully extract the intensional content from an affine function f, as a "strategy" — a cone of a sequence of $\Pi\Sigma$ orders approximating the function-space $A \multimap B$ — which computes it. We will now define "parallel composition plus hiding" of such strategies directly, and show that it corresponds to composition of the associated affine functions.

We define composition of strategies in a slightly more general setting based on the $⅋$ connective (as this can be used to represent the strict and non-strict function-spaces).

Given $\Pi\Sigma$-orders $A = \Sigma_{j \in J} A_i$ and $B = \Pi_{k \in K} \Sigma_{l \in L_k} B_{kl}$, define:

$$A⅋B = \Pi_{k \in K}((\Sigma_{j \in J_i}(A⅋\Sigma_{l \in L_k} B_{kl})) \oplus \Sigma_{l \in L_k}(\Sigma_{j \in J_i} A_{ij}⅋\Sigma_{l \in L_k} B_{kl}))$$

For each $\Pi\Sigma$ order, we write $\overline{A}$ for the "$\Sigma\Pi$-order" with products and sums swapped over — i.e. $\overline{\Pi_{i \in I}\Sigma_{j \in J_i} A_{ij}} = \Sigma_{i \in I}\Pi_{j \in J_i}\overline{A_{ij}}$ — so that $A \multimap B \cong \overline{A}⅋B$. For $\sigma \in A⅋B$, we write $\widehat{\sigma}$ for the element of $B⅋A$ obtained by symmetric relabelling.

Definition 4.21 *If $\sigma \in \overline{A}⅋B$ and $\tau \in \overline{B}⅋C$, then $\sigma|\tau : \overline{A}⅋C$ is defined recursively as follows: $\sigma|\top = \top$, $\sigma|\bot = \bot$.*
and $\sigma|\langle \tau_i \mid i \in I\rangle = \langle \sigma|\tau_i \mid i \in I\rangle$, where $\sigma|\mathtt{in_r}(\mathtt{in}_i(\tau)) = \mathtt{in}_i(\sigma|\tau)$ and $\langle \sigma_i \mid i \in I\rangle|\mathtt{in_l}(\mathtt{in}_j(\tau)) = \widehat{\widehat{\tau}|\widehat{\sigma_i}}$

Thus we can compute the composition of affine functions $f : A \to B$ and $g : B \to C$ by approximating them as strategies on the associated $\Pi\Sigma$ forms, applying Definition 4.21 and mapping back to $A \multimap C$.

Proposition 4.22 *For any affine functions $f : A \to B$ and $g : B \to C$, we have $f;g = \bigsqcup\{e^i(p^i(f);p^i(g)) \mid i \in \omega\}$.*

Proof: We prove this first for bounded A, B, C by induction on rank, following Proposition 4.11. It then follows for unbounded A, B by continuity. □

5 Linear Decomposition of Bistable Functions

We now complete our description of the locally boolean model of linear type theory by giving a *linear decomposition* of the category $\mathcal{LBB}$ of locally boolean domains and bistable and $\sqsubseteq$-continuous functions. By Proposition 2.8, every locally boolean domain is a bistable biorder; using the Decomposition Theorem, we may strengthen this result to show that every locally boolean domain is a *bistable bicpo* — a bistable biorder in which the extensional order is complete, and for any directed sets X, Y such that for all $x \in X$ and $y \in Y$ there exists $x' \in X$ and $y' \in Y$ such that $x \sqsubseteq x'$, $y \sqsubseteq y'$ and $x' \updownarrow y'$ we have $\bigsqcup X \updownarrow \bigsqcup Y$ and $\bigsqcup X \wedge \bigsqcup Y = \bigsqcup \{x \wedge y \mid x \in X \wedge y \in Y \wedge x \updownarrow y\}$ (see [11]). Hence $\mathcal{LBB}$ is a full subcategory of the category of bistable bicpos and bistable and $\sqsubseteq$-continuous functions; here we will establish that it is a Cartesian closed subcategory.

We show that $\mathcal{LBA}$ is a *reflective* subcategory of $\mathcal{LBB}$ — i.e. we will define a (monoidal) right adjoint, !, to the inclusion of $\mathcal{LBA}$ in $\mathcal{LBB}$, so that the bistable function-space $A \Rightarrow B$ decomposes as $!A \multimap B$. The intention is to define $!A = (A \Rightarrow \Sigma) \multimap \Sigma$, where $A \Rightarrow \Sigma$ is the cpo of continuous and bistable functions from A to Σ.

As in the affine and linear cases, we may define negation directly via its action on compact elements: $(\neg f)(x) = \bigsqcup \{\neg f(\neg p) \mid p \in CP(x)\}$, but to prove this correct, we show that it is the limit of a locally boolean chain of approximants.

Given elements $d \in A_k$ and $e \in \Pi_{i \in I-\{k\}} A_i$, let $e\lfloor d \rfloor_k \in \Pi_{i \in I} A_i$ be the k-*insertion* of d in e — i.e. $\pi_i(e\lfloor d \rfloor_k) = d$ if $i = k$, and $\pi_i(e\lfloor d \rfloor_k) = \pi_i(e)$ otherwise.

Lemma 5.1 *For any family of pointed lbds, $\{A_i \mid i \in I\}$,*

$$\Pi_{i \in I}(A_i)_\bot^\top \Rightarrow \Sigma \cong \Sigma_{k \in I}(A_i \times (\Pi_{i \in I-\{k\}}(A_i)_\bot^\top) \Rightarrow \Sigma)$$

Proof: Any monotone and bistable function into Σ is either constant or strict, and therefore by Lemma 4.8, k-strict for exactly one $k \in I$. Thus the required natural isomorphism sends each k-strict f to $\mathtt{in}_k(f')$, where $f'(\langle d, e \rangle) = f(e\lfloor \mathtt{in}(d) \rfloor_k)$. □

Lemma 5.2 *For any family of pointed lbos $\{A_{ij} \mid i \in I, j \in J_i\}$:*

$$\Pi_{i \in I} \Sigma_{j \in J_i} A_{ij} \Rightarrow \Sigma \cong \Sigma_{k \in I} \Pi_{l \in J_k}((A_{kl} \times \Pi_{i \in I-\{k\}} \Sigma_{j \in J_i} A_{ij}) \Rightarrow \Sigma)$$

Proof: By Lemma 5.1 $\Pi_{i \in I} \Sigma_{j \in J_i} A_{ij} \Rightarrow \Sigma$
$\cong \Sigma_{k \in I}((\coprod_{l \in J_k} A_{kl}) \times \Pi_{i \in I-\{k\}} \Sigma_{j \in J_i} A_{ij} \Rightarrow \Sigma)$
$\cong \Sigma_{k \in I}(\coprod_{l \in J_k}(A_{kl} \times \Pi_{i \in I-\{k\}} \Sigma_{j \in J_i} A_{ij}) \Rightarrow \Sigma)$
$\cong \Sigma_{k \in I} \Pi_{l \in J_k}((A_{kl} \times \Pi_{i \in I-k} \Sigma_{j \in J_i} A_{ij}) \Rightarrow \Sigma)$. □

We then use this lemma to show that $A \Rightarrow \Sigma$ is a limit of a locally boolean chain of $\Pi\Sigma$-orders — and hence a locally boolean domain.

Definition 5.3 *For each lbd $A = \Pi_{i \in I} \Sigma_{j \in J_i} A_{ij}$, we define a locally boolean ω-chain of $\Pi\Sigma$-orders — $\{(A \Rightarrow \Sigma)_i \mid i \in \omega\}$ as follows:*

$(A \Rightarrow \Sigma)_0 = *$
$(A \Rightarrow \Sigma)_{n+1} = \Sigma_{p \in \mathtt{g}(A)} \Pi_{x \in \mathtt{s}(A_p)}((A_{px} \times \Pi_{q \in \mathtt{g}(A)-\{p\}} A_q) \Rightarrow \Sigma)_n$.

Proposition 5.4 *$A \Rightarrow \Sigma$ is a limit for $\{(A \Rightarrow \Sigma)_i \mid i \in \omega\}$.*

PROOF: This follows the proof of Proposition 4.15 — using Lemma 5.2 we define an e-p pair $e^n_A : (A \Rightarrow \Sigma)_n \trianglelefteq A \Rightarrow \Sigma : p^n_A$ for each n, and show that for any $f \in A \to \Sigma$, $\bigsqcup_{n \in \omega} e^n_A(p^n_A(f)) = f$. We show by induction on $n = |x|$ that for all continuous functions $f : A \to \Sigma$ such that $f(x) = \top$ then $f^n(x) = \top$. $\square$

Corollary 5.5 *If A is a locally boolean domain then $A \Rightarrow \Sigma$ is a locally boolean domain.*

Hence $\_ \Rightarrow \Sigma$ acts as a functor from $\mathcal{LBB}$ to $\mathcal{PLBL}^{OP}$ with the following property.

Lemma 5.6 *$\_ \Rightarrow \Sigma : \mathcal{LBB} \to \mathcal{PLBL}^{OP}$ is left adjoint to $\_ \multimap \Sigma : \mathcal{PLBL}^{OP} \to \mathcal{LBB}$.*

PROOF: We have an an obvious natural isomorphism between $\mathcal{LBB}(A, B \multimap \Sigma)$, and $\mathcal{PLBL}(B, A \Rightarrow \Sigma)$. $\square$

By composing with the equivalence between $\mathcal{PLBL}^{OP}$ and $\mathcal{LBA}$ we obtain $! : \mathcal{LBB} \to \mathcal{LBA} = (\_ \Rightarrow \Sigma) \multimap \Sigma$, which is left adjoint to $(\_ \multimap \Sigma) \multimap \Sigma$, and hence also to the inclusion of $\mathcal{LBA}$ into $\mathcal{LB}$.

! is a symmetric monoidal functor; we have $!A \otimes !B = (((A \Rightarrow \Sigma) \multimap \Sigma) \multimap ((B \Rightarrow \Sigma) \multimap \Sigma) \multimap \Sigma) \multimap \Sigma \cong (((A \Rightarrow \Sigma) \multimap \Sigma) \multimap B \Rightarrow \Sigma) \multimap \Sigma \cong (A \Rightarrow B \Rightarrow \Sigma) \multimap \Sigma \cong (A \times B \Rightarrow \Sigma) \multimap \Sigma = !(A \times B)$.

Moreover, we have established that $\mathcal{LBB}$ is a Cartesian closed (full) subcategory of the category of bistable bicpos, since if A, B are locally boolean domains, then so is $A \Rightarrow B \cong !A \multimap B$. So this adjunction resolves a monoidal co-monad on $\mathcal{LBA}$, the co-Kleisli category of which is equivalent to $\mathcal{LBB}$.

6 Conclusions and Further Directions

As we have noted in the introduction, there are many other possible "intensional representations" of observably sequential, or bistable functions, including sequential algorithms [5,6] and strategies for different varieties of games [12,6,8]. We can relate bistable functions to these representations directly, as in notes by Streicher [14], Curien [7], and the author (we may observe, for example, that the strategies on Lamarche-style games [12] (AJM games without a bracketing condition [2]) with a single "error move" $\top$ form a locally boolean domain in which strategies containing a single, maximal length sequence are the prime elements). Alternatively, we may use the more abstract order-theoretic and categorical characterization of game trees as $\Pi\Sigma$ orders which has been developed here. We may, for example, easily show that the Lamarche-style games satisfy a version of the prenex normal form theorem: for any game A, we have $A \cong \Pi_{m\in M_A}\Sigma_{n\in P_m}A_{mn}$, where M_A is the set of (Opponent) moves of A, P_m is the set of moves n such that mn is a valid play, and A_{mn} is the game consisting of plays s such that $mn \cdot s$ is a play of A. Similarly, Proposition 4.11 holds for strategies in this category, and "parallel composition plus hiding" of strategies corresponds to composition of affine bistable functions via Proposition 4.22. Thus we can show that this category is equivalent to $\mathcal{PLBA}$, its coproduct completion is equivalent to $\mathcal{LBA}$, and the co-Kleisli category of the ! (equivalent to a CCC of sequential algorithms [6]) is equivalent to $\mathcal{LBB}$.

Work on identifying an extension of the "linear decomposition" of locally boolean domains described here to a more general class of bistable biorders, based on adding the stable order explicitly, is ongoing. The existence of complete and prime-algebraic bistable biorders which are not locally boolean (such as bistable bidomains with at least three elements, in which the bistable order is linear) poses the problem of how to describe bistable functions on such orders intensionally — in other words, how to specify them as games and strategies.

Unlike the bistable model of intuitionistic logic [3] (the simply-typed λ-calculus) our model of linear logic, like the corresponding games models, is *not* universal — there are various classes of elements which do not correspond to derivations in intuitionistic linear type theory or polarized MALL. This suggests two possible further questions. First — how can we characterise the definable elements or otherwise construct a universal model? Second — which programming languages or calculi can be given universal models without such constraints? We have addressed the former with standard techniques such as realizability, to obtain a fully complete model of polarized MALL, for example. Investigation of the latter question has led to bistable models of languages with state, and further control features such as coroutines, using categorical constructions described in [9]. These are based on the right-strict tensor, or "sequoidal" product, which may be defined on locally boolean domains as $(A \multimap (B \multimap \Sigma)) \multimap \Sigma$.

[3] To be precise, universality for logics without the "paraproofs" $\bot$ and $\top$ requires the restriction to *total* elements x such that $x = \neg x$.

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