CM20167: Why equating all divergent terms in the λ-calculus leads to inconsistency

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Suppose we have a theory of equality of λ-terms, $=$, such that:

- $=$ is an equivalence relation
- $=$ contains $=_\beta$ (that is, if $M =_\beta N$ then $M = N$),
- $=$ is closed under term formation, i.e. if $M = N$ then $MP = NP$, $PM = PN$ and $\lambda x.M = \lambda x.N$ for all x and P

Lemma 1 If the theory equates the terms $S \equiv \lambda xyz.xz(yz)$ and $K \equiv \lambda xy.x$ then it is inconsistent, i.e. it equates any two terms.

Proof Since $S = K$ we also have $SABC = KABC$ for any terms A, B and C (by the second condition above) and since $SABC =_\beta AC(BC)$ and $KABC =_\beta AC$ we get $AC(BC) = AC$.

Now suppose $A \equiv C \equiv \lambda x.x$. Then the above equation gives us $B(\lambda x.x) = \lambda x.x$.

Let $B = KD$ for any D. Then $B(\lambda x.x) =_\beta D$ so we get $D = \lambda x.x$ for any D, which means every pair of terms is equated, i.e. $=$ is inconsistent. ■

Now we can prove that if $=$ equates every pair of terms M and N which have no normal form, then it is inconsistent.

Theorem 2 If $M = N$ for any pair of terms M, N such that M and N have no normal form, then $=$ is inconsistent.

Proof Let $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, which has no normal form. It is clear that $\lambda x.xK\Omega$ has no normal form, and neither does $\lambda x.xS\Omega$.

So our theory equates these terms, and therefore also

$$(\lambda x.xK\Omega)K = (\lambda x.xS\Omega)K.$$

But $(\lambda x.xK\Omega)K \to_\beta KK\Omega \to_\beta^* K$ and $(\lambda x.xS\Omega)K \to_\beta KS\Omega \to_\beta^* S$, so we get $K = S$. Then by our lemma, the theory is inconsistent. ■