Some introductory exercises on the λ-calculus

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1. For each of the terms below, draw an arrow from every bound occurrence of a variable to its binding occurrence. Circle every free occurrence.

   (a) x

   (b) $\lambda x.x$

   (c) $(\lambda a.z)a$

   (d) $(\lambda n.n)z$

   (e) $\lambda z.(\lambda y.(\lambda x.x)y)z$

   (f) $(\lambda t.((\lambda t.(\lambda t.t)t)t))t$

2. Show that if $x \notin \mathsf{FV}(M)$ then $M[N/x] \equiv M$.

3. For each of the terms below, perform one step of β-reduction, if there is one to perform.

   (a) $\lambda x.x$

   (b) $x(\lambda x.x)(\lambda y.y)$

   (c) $(\lambda x.x)(\lambda y.y)$

   (d) $(\lambda x.xx)(\lambda y.yy)$

   (e) $x((\lambda x.xy)z)y$

   (f) $(\lambda x.xy)xy$

   (g) $(\lambda x.(\lambda y.xy))y$

   (h) $(\lambda x.(\lambda y.xy))(\lambda y.y)$

   (i) $(\lambda x.(\lambda y.xy)x)z$

   (j) $(\lambda x.y)(\lambda x.xx)(\lambda x.xx)$

   (k) $(\lambda x.y)((\lambda x.xx)(\lambda x.xx))$

   (l) $(\lambda x.xxx)(\lambda x.xxx)$

   (m) $(\lambda xy.yx)(\lambda z.z)$

   (n) $(\lambda xy.yxy)(\lambda xy.yxy)(\lambda xy.yxy)$

   (o) SKK, where $S = \lambda xyz.xz(yz)$ and $K = \lambda xy.x$

(p) $Y_1 Y_1 M$, where $Y_1 = \lambda f.\lambda x.x(ffx)$ and M is any λ-term.

Once you've done the first reduction on each of these, perform some more reductions until you reach a normal form, or convince yourself there is no normal form.

4. The term W is defined to be

$$\lambda x.\lambda y.xyy.$$

Give a complete analysis of the sequences of reductions that the term WWW can perform.

5. Show that $=_\alpha$ is an equivalence relation, that is, that

- $M =_\alpha M$ for every term M
- if $M =_\alpha N$ then $N =_\alpha M$.
- if $M =_\alpha N$ and $N =_\alpha P$ then $M =_\alpha P$.