CM10196: Discrete Mathematics for Computation
Problem Sheet 7

Set November 23rd 2007; hand in by Dec 6th 2007

Coursework forms 25% of the assessment for this unit. Coursework will consist of your answers to eight problem sheets, plus the "learning log" exercise. Each problem sheet will be marked out of 10, and there will be 20 marks for the learning log.

On this sheet, each question is worth two marks.

1. Below we define four binary relations on the set $\mathbb{Z}$ of integers. For each relation, say whether or not it is a function and justify your answers.
   (a) $R_1 = \{(x, y) \mid x = y + 2\}$.
   (b) $R_2 = \{(x, y) \mid x = y^2\}$.
   (c) $R_3 = \{(x, y) \mid x^2 = y\}$.
   (d) $R_4 = \{(x, y) \mid x = y\}$.
2. (a) Prove that if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are injective then so is $g \circ f$.
   (b) Give an example of functions f and g which are not both injective, such that $g \circ f$ is injective.
3. Let A, B, C and D be sets, and suppose that $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$ are functions.
   (a) Show that $h \circ (g \circ f) = (h \circ g) \circ f$.
   (b) What do you think this property is called?
4. Suppose A and B are finite sets, and that B has fewer elements than A (i.e. $|A| > |B|$).
   (a) Let $f : A \longrightarrow B$ be a function. Prove that f is not injective.
   (b) Let $g : B \longrightarrow A$ be a function. Prove that g is not surjective.

Now suppose C and D are finite sets and $h_1 : C \longrightarrow D$ and $h_2 : D \longrightarrow C$ are both injective functions. What can you conclude about the sizes of C and D? What if instead of being injective, both h_1 and h_2 are surjective?

5. Let A be any set.

   (a) Show that the identity function $\mathsf{id} : A \longrightarrow A$ has the property that, for any set B and function $f : A \longrightarrow B$, $f \circ \mathsf{id} = f$, and for any function $g : B \longrightarrow A$, $\mathsf{id} \circ g = g$.

   (b) Show that no other function from A to A has this property. Hint: suppose there are two functions with this property, and see what happens when you compose them together.