Topic 1: Propositional logic

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Logic!

- This lecture is about the simplest kind of mathematical logic: *propositional calculus.*
- We discuss *propositions*, which are statements that can be either *true* or *false*.
- Every proposition is either true or false; there is no middle ground.

Propositions

This view of propositions is rather simple minded. Is it reasonable for the following statements?

- $2 + 2 = 4$.
- $3 * 5 = 23948723894$.
- it is raining.
- Bath is a beautiful city.

Connectives, laws, ...

We will study the logic of these simple propositions by considering:

connectives	ways of combining propositions
laws	facts about propositions that hold regardless of the truth values of those propositions
tautologies	propositions that are always true, no matter what the world looks like.

Proofs

A central theme of mathematical logic is that of proofs. Important questions include:

- what constitutes a valid proof?
- can we be sure that things we prove are actually true?
- can we prove everything that is true?

We will not consider these ideas much in this course.

Conjunction and disjunction

The two most basic connectives are conjunction (AND) and disjunction (OR). In symbols, conjunction is $\wedge$, while disjunction is $\vee$.
Given propositions x and y, we can form new propositions

$$x \wedge y \qquad\qquad x \vee y.$$

$x \wedge y$ is true if both x and y are; it is false otherwise.

$x \vee y$ is true if either x or y is true, or if both are true; it is false otherwise.

Truth tables

This kind of definition is often expressed as a *truth table* (cf. multiplication tables).

Truth table for $\wedge$:

x	y	$x \wedge y$
t	**t**	**t**
t	**f**	**f**
f	**t**	**f**
f	**f**	**f**

And for $\vee$:

x	y	$x \vee y$
t	**t**	**t**
t	**f**	**t**
f	**t**	**t**
f	**f**	**f**

Negation

As well as talking about a statement, such as "Angela is Bens sister", we can talk about its negation: "Angela is not Bens sister".

The negation of a proposition x is written $\neg x$ and pronounced "not x".

It has an easy truth table:

x	$\neg x$
t	**f**
f	**t**

Implication

Implication is the connective corresponding to if... then...

Here are some examples... which ones are true?

- if $n > m$ then $n + 1 > m + 1$.
- if $n > m$ then $2 + 2 = 4$.
- if $2 + 2 = 3$ then I'm a Dutchman.
- if $2 + 2 = 4$ then I'm a Dutchman.

Truth of implication

An implication, written $\longrightarrow$, says that if something (the antecedent) is true, then something else (the consequent) is also true.

It says nothing about what happens if the antecedent is false.

Therefore, the only way an implication can be false is

- the antecedent must be true, so that the implication statement has something to say;
- the consequent must be false, so that what the implication statement is saying is false.

Truth table for implication

Truth table for $\longrightarrow$:

x	y	$x \longrightarrow y$
t	**t**	**t**
t	**f**	**f**
f	**t**	**t**
f	**f**	**t**

Boolean formulae

We will now take a very simple, but very common mathematical step.

We've defined a collection of *operations* ($\wedge$, $\vee$, $\neg$, $\longrightarrow$) which operate on a collection of *values* (**t**, **f**).

We now begin to talk about *formulae*, in which we use *variables* like X and Y to stand for unknown values, and combine them with the operations.

This is just like the beginnings of algebra in school: we can start to talk about formulae like $X + X = 2 \times X$ and wonder whether they are true or not. . .

Boolean formulae

Here's a formal definition.

Definition

Given a collection of variables, $X_1 \dots X_n$, a *Boolean formula* is a string of symbols from the set

$$\{X_1, \dots, X_n, \wedge, \vee, \neg, \rightarrow, (,)\}$$

of the following form:

- any variable X_i is a Boolean formula
- if F and G are Boolean formulae, so are $(F \wedge G)$, $(F \vee G)$, $(F \rightarrow G)$ and $\neg F$.

Boolean formulae

That definition is extremely formal.

All it says is that we build Boolean formulae by starting with variables and combining them with the operations $\wedge$, $\vee$ etc., adding brackets to keep things unambiguous.

In practice we will often not bother to write all the brackets.

Boolean formulae

Which of these are Boolean formulae?

- $(X \wedge Y)$
- $(X \neg Y)$
- $(X \wedge (Y \vee Z))$
- $(X \rightarrow)$
- $\neg X$
- $((X \rightarrow \neg Y) \vee (\neg Z \wedge Y))$

Evaluating Boolean formulas

Given a Boolean formula containing variables $X_1, \ldots, X_n$, we can *evaluate* that formula for any given truth values of those variables.
If we're told whether each X_i is true or false, we can

- replace each X_i in the formula by **t** or **f** appropriately
- work out the truth value of the whole formula using truth tables.

Evaluating Boolean formulas

Different truth values for the variables will give different truth values for the whole formula.

For instance, the formula $(X \rightarrow \neg Y) \vee (\neg Z \wedge Y)$ will evaluate to **t** if we set

X	**t**
Y	**t**
Z	**f**

but to **f** if we set

X	**t**
Y	**t**
Z	**t**

Evaluating a formula

Let's evaluate $(X \rightarrow \neg Y) \vee (\neg Z \wedge Y)$ with X, Y and Z all set to **t**.

$$
\begin{array}{ccccccc}
(X & \rightarrow & \neg Y) & \vee & (\neg Z & \wedge & Y) \\
(\mathbf{t} & \rightarrow & \neg \mathbf{t}) & \vee & (\neg \mathbf{t} & \wedge & \mathbf{t}) \\
(\mathbf{t} & \rightarrow & \mathbf{f}) & \vee & (\neg \mathbf{t} & \wedge & \mathbf{t}) \\
 & \mathbf{f} & & \vee & (\neg \mathbf{t} & \wedge & \mathbf{t}) \\
 & \mathbf{f} & & \vee & (\mathbf{f} & \wedge & \mathbf{t}) \\
 & \mathbf{f} & & \vee & & \mathbf{f} & \\
 & & & \mathbf{f} & & &
\end{array}
$$

Tautologies

Some formulas are true regardless of what truth values their variables take. These are called *tautologies*.

Definition

A Boolean formula F with variables $X_1, \ldots, X_n$ is called a *tautology* if the value of F is $\mathbf{t}$ for any assignment of true/false values to the variables X_i.

Example

The following are tautologies:

- $X \vee \neg X$
- $\neg\neg X \to X$
- $X \to \neg\neg X$
- $\neg(X \wedge \neg X)$
- $((\neg X \to Y) \wedge (\neg X \to \neg Y)) \to X$.

Checking that $X \vee \neg X$ is a tautology

To check that $X \vee \neg X$ is a tautology, we have to check that setting X to $\mathbf{t}$ makes the formula true, and that setting X to $\mathbf{f}$ makes the formula true.
Setting X to $\mathbf{t}$:

$$\begin{array}{ccc} (X & \vee & \neg X) \\ (\mathbf{t} & \vee & \neg\mathbf{t}) \\ (\mathbf{t} & \vee & \mathbf{f}) \\ & \mathbf{t} & \end{array}$$

Setting X to $\mathbf{f}$:

$$\begin{array}{ccc} (X & \vee & \neg X) \\ (\mathbf{f} & \vee & \neg\mathbf{f}) \\ (\mathbf{f} & \vee & \mathbf{t}) \\ & \mathbf{t} & \end{array}$$

Exercise

Now check that the other examples from a few slides back are all tautologies.

Since most of them involve only the variable X, there are only two cases to consider: when X is $\mathbf{t}$ and when X is $\mathbf{f}$.

For the last one, $((\neg X \to Y) \wedge (\neg X \to \neg Y)) \to X$, there are two variables and hence four cases.

However, remember that an implication is always true if the right hand side (the consequent) is true, so in practice you only need to worry about the cases when X is set to $\mathbf{f}$.

Make sure you understand why!

Substituting in tautologies

A tautology like $X \vee \neg X$ is true regardless of what X is.

This means that we can replace X by any proposition, or any other formula, and we still have a tautology.

For example, replacing X by $Y \rightarrow Z$ gives us

$$(Y \rightarrow Z) \vee \neg(Y \rightarrow Z)$$

which is still a tautology.

Logical equivalence

A pair of formulas F_1 and F_2 are *logically equivalent* if one holds whenever the other does.

That is to say:

- if F_1 is true then so is F_2 i.e. $F_1 \rightarrow F_2$
- if F_2 is true then so is F_1 i.e. $F_2 \rightarrow F_1$.

That is to say, F_1 and F_2 are equivalent if

$$(F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

is a tautology.

This formula is so important we introduce special notation for it: we write $F_1 \equiv F_2$ as an abbreviation for the above.

Laws

Mathematicians like to talk about *laws* that various operators obey, for instance:

Commutativity	Addition and multiplication are commutative: $3 + 4 = 4 + 3$, $2 \times 9 = 9 \times 2$ etc. In general, $x + y = y + x$ and $x \times y = y \times x$.
Associativity	When doing more than one addition or multiplication, the order in which you do the operations does not matter: $3 \times (4 \times 5) = (3 \times 4) \times 5$ and so on. What's the general form?
Distributivity	There are laws that let you "distribute" one operation over another: $3 \times (4 + 5) = (3 \times 4) + (3 \times 5)$. What's the general form?

Laws of logic

The basic laws of the operations $\wedge$ and $\vee$ are very much like those of $\times$ and $+$: the operations are commutative and associative, and there's a distributive law for $\wedge$ over $\vee$. (Unlike in arithmetic, there's another distributive law too.)

Commutativity of $\wedge$ and $\vee$

Let x and y be propositions. Then

- $x \wedge y$ is true if and only if $y \wedge x$ is true.
- $x \vee y$ is true if and only if $y \vee x$ is true.

We could also phrase this by saying: given variables X, Y, the formulas

- $X \wedge Y \equiv Y \wedge X$
- $X \vee Y \equiv Y \vee X$

are tautologies.

Associativity

Associativity of $\wedge$

For variables X, Y and Z, the formula

$$X \wedge (Y \wedge Z) \equiv (X \wedge Y) \wedge Z$$

is a tautology.

Associativity of $\vee$

For variables X, Y and Z, the formula

$$X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z$$

is a tautology.

Distributive laws

In arithmetic, we have the distributive law

$$x \times (y + z) = (x \times y) + (x \times z).$$

In logic there are two similar laws:

Distribution of $\wedge$ over $\vee$

Given variables X, Y and Z, the formula

$$X \wedge (Y \vee Z) \equiv (X \wedge Y) \vee (X \wedge Z)$$

is a tautology.

Distribution of $\vee$ over $\wedge$

Given variables X, Y and Z, the formula

$$X \vee (Y \wedge Z) \equiv (X \vee Y) \wedge (X \vee Z)$$

is a tautology.

Easy exercises

1. Prove that all these laws hold by evaluating the appropriate truth tables.
2. We have one distributive law for $+$ and $\times$ and two for $\wedge$ and $\vee$. What would the second law for $+$ and $\times$ look like? Why is it not true?

De Morgan's Laws

The *de Morgan laws* relate $\wedge$ and $\vee$ via $\neg$:

de Morgan's laws

Given variables X and Y, the formulae

$$\begin{aligned} \neg(X \wedge Y) &\equiv (\neg X \vee \neg Y) \\ \neg(X \vee Y) &\equiv (\neg X \wedge \neg Y) \end{aligned}$$

are tautologies.

You should check this!

More laws of logic

1. $X \vee \neg X$. The law of *excluded middle*.
2. $\neg\neg X \equiv X$.
3. $\neg(X \wedge \neg X)$.
4. $((\neg X \to Y) \wedge (\neg X \to \neg Y) \to X$. *Reductio ad absurdum*, or proof by contradiction.
5. $(X \to Y) \equiv (\neg X \vee Y)$.

Check all these!

What do the laws mean?

The law $X \vee \neg X$ says that, whatever proposition you're thinking of (X), either it's true, or its negation is true.

Put another way, *every proposition is either true or false*.

Put another way, *there is no middle ground between true and false*. That's why it's called the *law of excluded middle*.

What do the laws mean?

The law $((\neg X \rightarrow Y) \wedge (\neg X \rightarrow \neg Y)) \rightarrow X$ is called *proof by contradiction*.

It is used as follows. We want to prove that something (X) is true.
Instead of showing that X is true, we show that it *cannot be false*.

To do this, show that if $\neg X$ holds then so do two contradictory statements: Y and $\neg Y$. Since that is impossible, $\neg X$ cannot be true, so X must be.

A proof by contradiction

Theorem

There are infinitely many prime numbers.

Proof Suppose there are finitely many prime numbers. Then for some numner n there are exactly n primes. Call these $p_1, \ldots, p_n$.

Consider the number

$$(p_1 \times p_2 \times \cdots \times p_n) + 1.$$

It's easy to see that this does not divide by any of the primes $p_1, \ldots, p_n$.
It is therefore a prime. So there are more than n primes.

We have shown that there are both n primes and more than n primes, a contradiction.

So our assumption must be false, i.e. there are infinitely many primes. ☐

Doing without connectives

The law $(X \to Y) \equiv (\neg X \vee Y)$ means that any formula involving $\to$ can be rewritten to an equivalent one using $\neg$ and $\vee$.

This means that, if we want to, we can always do without $\to$ and just use $\neg$ and $\vee$ instead.

Similarly, we have

$$\begin{aligned} X \wedge Y &\equiv \neg\neg(X \wedge Y) \\ &\equiv \neg(\neg X \vee \neg Y) \end{aligned}$$

so we can replace $\wedge$ with $\neg$ and $\vee$ too.

This means that any boolean formula can be equivalently written using just $\neg$ and $\vee$.

Doing without connectives

Alternatively, we could use $\neg$ and $\wedge$ to express $\vee$, so we can write any formula just using $\neg$ and $\wedge$.

Similarly, $\neg$ and $\to$ would do.

In Boolean circuits, there's an operation called **NAND**, which is short for "not and". It's defined by

$$x \ \mathbf{NAND}\ y = \neg(x \wedge y).$$

It's important because it turns out that you can write everything just in terms of **NAND**.

exercise

Show how to express $\wedge$, $\vee$, $\neg$ and $\to$ using just **NAND**.

Disjunctive normal form

We're now going to show that every Boolean formula can be written in a particular special shape, called a *normal form*.

This is helpful for various reasons. One is that it gives you a handle on what kinds of formulas there are. Another is that it makes it relatively easy to design circuits to implement formulas.

Definition

A formula is in *disjunctive normal form* (DNF) if it is of the form

$$F_1 \vee F_2 \vee \cdots \vee F_n$$

where each F_i is of the form

$$L_1 \wedge L_2 \wedge \cdots \wedge L_{k_i}$$

and each L_i is either a variable, X, or the negation of a variable, $\neg X$.

Disjunctive normal form

Example

The following formulas are in disjunctive normal form:

- $X \vee (Y \wedge \neg Z)$.
- $X \wedge \neg Y$.
- X.
- $(X \wedge \neg Y) \vee (Y \wedge Z)$.

The following are not in disjunctive normal form:

- $X \wedge (Y \vee Z)$.
- $\neg(X \wedge Y)$.
- $X \vee \neg\neg X$.

DNFs as truth tables

DNFs are just a kind of "truth table" written as a formula.
A formula like $(X \rightarrow \neg Y) \vee (\neg Z \wedge Y)$ can be thought of as having a truth table:

X	Y	Z	$(X \rightarrow \neg Y) \vee (\neg Z \wedge Y)$
t	**t**	**t**	**f**
t	**t**	**f**	**t**
t	**f**	**t**	**t**
t	**f**	**f**	**t**
f	**t**	**t**	**t**
f	**t**	**f**	**t**
f	**f**	**t**	**t**
f	**f**	**f**	**t**

DNFs as truth tables

This truth table tells us that the formula is true if the values of the variables are those on any of the lines apart from the top one.

The second line has X and Y true, and Z false. This happens exactly when

$$X \wedge Y \wedge \neg Z$$

holds.

Similarly, the third line comes into play when

$$X \wedge \neg Y \wedge Z$$

holds.

We can write similar formulae expressing each of those seven lines.

The truth table tells us that, if line 2 holds or line 3 holds or line 4 holds or line 5 holds or line 6 holds or line 7 holds or line 8 holds, then the formula is true, and otherwise it is false.

DNFs as truth tables

That means that the formula is true exactly when

$$\begin{array}{ll} & (X \wedge Y \wedge \neg Z) \\ \vee & (X \wedge \neg Y \wedge Z) \\ \vee & (X \wedge \neg Y \wedge \neg Z) \\ \vee & (\neg X \wedge Y \wedge Z) \\ \vee & (\neg X \wedge Y \wedge \neg Z) \\ \vee & (\neg X \wedge \neg Y \wedge Z) \\ \vee & (\neg X \wedge \neg Y \wedge \neg Z) \end{array}$$

which is a DNF!

We can do the same trick to generate a DNF for any formula.

The DNF Theorem

Theorem

Every Boolean formula is equivalent to one in DNF.

Proof Do the truth table trick outlined above. ☐

Simpler DNFs

The DNF we just created is not as simple as it could be.

$$\begin{array}{ll} & (X \wedge Y \wedge \neg Z) \\ \vee & (X \wedge \neg Y \wedge Z) \\ \vee & (X \wedge \neg Y \wedge \neg Z) \\ \vee & \color{red}{(\neg X \wedge Y \wedge Z)} \\ \vee & \color{red}{(\neg X \wedge Y \wedge \neg Z)} \\ \vee & (\neg X \wedge \neg Y \wedge Z) \\ \vee & (\neg X \wedge \neg Y \wedge \neg Z) \end{array}$$

The two lines in red, when taken together, are equivalent to just

$$\neg X \wedge Y.$$

Simpler DNFs

This gives us

$$\begin{array}{ll} & (\neg X \wedge Y) \\ \vee & (X \wedge Y \wedge \neg Z) \\ \vee & (X \wedge \neg Y \wedge Z) \\ \vee & (X \wedge \neg Y \wedge \neg Z) \\ \vee & {\color{red}(\neg X \wedge \neg Y \wedge Z)} \\ \vee & {\color{red}(\neg X \wedge \neg Y \wedge \neg Z)} \end{array}$$

Again we can simplify the red lines to just $\neg X \wedge \neg Y$.

Simpler DNFs

This gives us

$$\begin{array}{ll} & {\color{red}(\neg X \wedge Y)} \\ \vee & {\color{red}(\neg X \wedge \neg Y)} \\ \vee & (X \wedge Y \wedge \neg Z) \\ \vee & (X \wedge \neg Y \wedge Z) \\ \vee & (X \wedge \neg Y \wedge \neg Z) \end{array}$$

Again we can simplify the red lines to just $\neg X$.

Simpler DNFs

This gives us

$$\begin{array}{ll} & \neg X \\ \vee & (X \wedge Y \wedge \neg Z) \\ \vee & {\color{red}(X \wedge \neg Y \wedge Z)} \\ \vee & {\color{red}(X \wedge \neg Y \wedge \neg Z)} \end{array}$$

Now the red lines simplify to just $X \wedge \neg Y$.

Simpler DNFs

This gives us

$$\begin{array}{ll} & \neg X \\ \vee & (X \wedge \neg Y) \\ \vee & (X \wedge Y \wedge \neg Z) \end{array}$$

Since the first line handles all the cases where X is not true, we can drop all the Xs in later lines. This gives us:

$$\begin{array}{ll} & \neg X \\ \vee & \neg Y \\ \vee & (Y \wedge \neg Z) \end{array}$$

Simpler DNFs

$$\begin{array}{ll} & \neg X \\ \vee & \neg Y \\ \vee & (Y \wedge \neg Z) \end{array}$$

Similarly, the second line handles all the cases where Y is not true, so we can drop Y from the last line giving us just

$$\neg X \vee \neg Y \vee \neg Z$$

which is a lot simpler!

Calculating DNFs using laws

We don't have to go through the truth table to produce a DNF. Often it's easier just to use the laws that we know:

$$\begin{array}{ll} & (X \rightarrow \neg Y) \vee (\neg Z \wedge Y) \\ \equiv & (\neg X \vee \neg Y) \vee (\neg Z \wedge Y) \\ = & \neg X \vee \neg Y \vee (\neg Z \wedge Y) \end{array}$$

which is a DNF. Again we can simplify it to $\neg X \vee \neg Y \vee \neg Z$ if we want to.

Conjunctive normal form

There's another kind of normal form called *conjunctive normal form* which is sometimes useful.

Definition

A formula is in *conjunctive normal form* (CNF) if it is of the form

$$F_1 \wedge F_2 \wedge \cdots \wedge F_n$$

where each F_i is of the form

$$L_1 \vee L_2 \vee \cdots \vee L_{k_i}$$

and each L_i is either a variable, X, or the negation of a variable, $\neg X$.

The CNF theorem

Theorem

Every Boolean formula is equivalent to one in CNF.

Proof Let B be a boolean formula. By the DNF theorem, $\neg B$ it is equivalent to a DNF $F_1 \vee \cdots \vee F_n$. Therefore B is equivalent to $\neg(F_1 \vee \cdots \vee F_n)$.

By de Morgan's law,

$$\neg(F_1 \vee \cdots \vee F_n) \equiv \neg F_1 \wedge \cdots \wedge \neg F_n.$$

Each F_i is $L_1 \wedge \cdots \wedge L_{k_i}$ so again by de Morgan's law, each $\neg F_i$ is equivalent to

$$\neg L_1 \vee \cdots \vee \neg L_{k_i}.$$

Each $\neg L_j$ is either $\neg X$ or $\neg\neg X$ which is equivalent to X, so this gives us a CNF equivalent to B. $\square$