A Posetal Cartesian Doubly-Closed Category

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Abstract

We describe a poset which has the structure of a Cartesian doubly-closed category, and is therefore a categorical model of **BI**, O'Hearn and Pym's Logic of Bunched Implications [1], albeit a very degenerate one.

1 A posetal CDCC

Consider the poset $(\mathbb{N}, \geq)$, i.e. the natural numbers ordered by the greater-than relation. We shall show that this has two monoidal-closed structures, one of which is cartesian.

For the first, consider the monoid operation $+$. This is clearly a monoid on $\mathbb{N}$. To see that it has a right adjoint, consider that

$$\begin{array}{ll} & a + b \geq c \\ \iff & a \geq c - b \\ \iff & a \geq (c - b) \sqcup 0 \end{array}$$

since $a \geq 0$ for any $a \in \mathbb{N}$. Therefore the operation taking c to $(c - b) \sqcup 0$ is right adjoint to the operation taking a to $a + b$.

For the cartesian structure, there is of course only one choice of monoid operation: the maximum operator. It is a product since

$$\begin{array}{ll} & a \geq b \sqcup c \\ \iff & (a \geq b) \wedge (a \geq c) \end{array}$$

Finally we demonstrate that it has a right adjoint:

$$\begin{array}{ll} & a \sqcup b \geq c \\ \iff & (a \geq c) \vee (b \geq c) \\ \iff & (a \geq c \wedge c > b) \vee (a \geq 0 \wedge b \geq c) \end{array}$$

so we can define

$$b \Rightarrow c = \begin{cases} c & \text{if } c > b \\ 0 & \text{otherwise} \end{cases}$$

and we then have $a \sqcup b \geq c \iff a \geq b \Rightarrow c$.

2 Constructing non-trivial CDCCs

We can now use our posetal CDCC to produce non-trivial CDCCs from Cartesian closed categories: since for any two CCCs $\mathbf{C}$ and $\mathbf{D}$ it is the case that $\mathbf{C} \times \mathbf{D}$ is a CCC, and similarly for SMCCs, we have:

Theorem 1 For any CCC $\mathbf{C}$, the category $\mathbf{C} \times \mathbb{N}$, where $\mathbb{N}$ is ordered by greater-than, is a CDCC.

References

[1] P. W. O'Hearn and D. J. Pym. The logic of bunched implications. *Bulletin of Symbolic Logic*, 5(2):215–244, June 1999.