

# Reasoning about Idealized ALGOL Using Regular Expressions

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**Abstract.** We explain how recent developments in games semantics can be applied to reasoning about equivalences of terms in a non-trivial fragment of Idealized ALGOL (IA). Being derived directly from the fully abstract games semantics for IA, our method of reasoning preserves its desirable theoretical properties. The method is mathematically elementary and formal, which makes it uniquely suitable for automation. We show that reasoning can be carried out using only an extended language of regular expressions, a language for which equivalence is formally decidable.

## 1 Introduction

Reynolds's *Idealized Algol* (IA) is a compact language which combines the fundamental features of procedural languages with a full higher-order procedure mechanism. This combination makes the language very expressive. For example, simple forms of classes and objects may be encoded in IA [15]. For this reason, IA has attracted a great deal of attention from theoreticians; some 20 papers spanning almost 20 years of research were recently collected in book form [9].

A common theme in the literature on semantics of IA, beginning with [4], is the use of models to validate certain equivalences between programs. These example equivalences are intended to capture intuitively valid principles such as the privacy of local variables, irreversibility of state-changes and so on. A good model should support these intuitions.

A frustrating situation was created with the development of a fully abstract game semantics for IA [1]. The full abstraction result means that the model validates all correct equivalences between programs, but unfortunately the model as originally presented is complicated, and calculating and reasoning within the model is difficult.

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In this paper, we show that if one restricts attention to the second-order subset of IA, the games model can be simplified dramatically: terms now denote regular languages, and a relatively straightforward notation can be used to describe and calculate with the simplified semantics. The fragment of IA which we consider contains almost all the example equivalences from the literature, and we are therefore able to validate them in a largely calculational, algebraic style, using our semantics. We also obtain a decidability result for equivalence of programs in this fragment.

We believe our new presentation of game semantics is elementary enough to be considered a potential “popular semantics” [17]; it should at least provide a point of entry to game semantics for those who have previously found the subject opaque. Moreover, the property of full abstraction together with the fact that reasoning can be carried out in a decidable formal language suggest that our approach constitutes a good foundation on which an automatic program checker for IA and related languages can be constructed.

## 2 The IA Fragment

The principles of the programming language IA were laid down by John Reynolds in an influential paper [16]. IA is a language that combines imperative features with a procedure mechanism based on a typed call-by-name lambda calculus; local variables obey a stack discipline, having a life time dictated by syntactic scope; expressions, including procedures returning a value, cannot have side effects, *i.e.* they cannot assign to variables. We conform to these principles, except for the last one. This flavour of IA is known as IA with *active expressions* and has been analysed extensively [19, 1, 7]. We consider only the recursion-free second order fragment of this language, the fragment which has been used to give virtually all the significant equivalences mentioned in the literature.

The data types of the language (*i.e.* types of data assignable to variables) are integers and booleans:

$$\tau ::= \mathbf{int} \mid \mathbf{bool}$$

The phrase types of the language are those of commands, variables and expressions, plus function types.

$$\sigma ::= \mathbf{comm} \mid \mathbf{var}[\tau] \mid \mathbf{exp}[\tau], \quad \theta ::= \sigma \mid \sigma \rightarrow \theta$$

Note that we include only first-order function types here. We will consider only terms of the form

$$\iota_1 : \theta_1, \dots, \iota_k : \theta_k \vdash M : \sigma$$

that is, terms of ground type with free variables of arbitrary first-order type. For the sake of simplicity in this paper, we also assume that  $M$  is  $\beta$ -normal, so that it contains no  $\lambda$ -abstractions and the only use of function application is applying a free identifier  $\iota$  to a series of arguments; this last restriction can be removed at the expense of a little notational overhead in the semantics.

The terms of the language are as follows. In type **comm** there are basic commands **skip**, to do nothing, and  $\Omega$  to diverge; in type **exp[int]** there are constants  $n$  for the integers; and in type **exp[bool]** there are the constants **true** and **false**. There are term formers for assignment to variables,  $V := E$ , dereferencing variables,  $!V$ , sequential composition of commands  $C; C'$ , and sequential composition of a command with an expression to yield a side-effecting expression  $C; E$ . We have a conditional operation **if**  $B$  **then**  $C$  **else**  $C'$ , a while-loop **while**  $B$  **do**  $C$ , application of first-order identifiers to arguments  $\iota M_1 \dots M_k$ , and the local-variable declaration **new** $[\tau]$   $\iota$  **in**  $C$ . Here, the free variable  $\iota : \mathbf{var}[\tau]$  of  $C$  becomes bound. Finally, we assume the usual range of binary operations on integer and boolean expressions.

### 3 Game Semantics of Idealized Algol

This section provides a brief account of the ideas behind the games model of Idealized Algol. Familiarity is not a necessity provided the reader is prepared to take our claim of full abstraction on trust.

In game semantics, a computation is represented as an *interaction* between two protagonists: Player (P) represents the program, and Opponent (O) represents the environment or context in which the program runs. For example, for a program of the form

$$\iota : \mathbf{exp[int]} \rightarrow \mathbf{comm} \vdash M : \mathbf{comm}$$

Player will represent the program  $M$ ; Opponent represents the context, in this case the non-local procedure  $\iota$ . This procedure, if called by  $M$ , may in turn call an argument, in which case O will ask P to provide this information.

The interaction between O and P consists of a sequence of moves, alternating between players. In the game for the type **comm**, for example, there is an initial move *run* to initiate a command, and a single response *done* to signal termination. Thus a simple interaction corresponding to the command **skip** might be

O: *run* (start executing)  
P: *done* (immediately terminate).

In more interesting games, such as the one used to interpret programs like

$$\iota : \mathbf{exp[int]} \rightarrow \mathbf{comm} \vdash \iota(0) : \mathbf{comm}$$

there are more moves. Corresponding to the result type **comm**, there are the moves *run* and *done*. The program needs to run the procedure  $\iota$ , so there are also moves  $run_\iota$  and  $done_\iota$  to represent that; here the  $run_\iota$  move is a move for P, and  $done_\iota$  is a move for O.

Finally, the procedure  $\iota$  may need to evaluate its argument. For this purpose, O has a move  $q_\iota^1$ , meaning “what is the value of the first argument to  $\iota$ ?”, to which P may respond with an integer  $n$ , tagged as  $n_\iota^1$  for identification’s sake.

Here is a sample interaction in the interpretation of the above term.

O: *run* (start executing)  
P: *run<sub>ι</sub>* (execute *ι*)  
O: *q<sub>ι</sub><sup>1</sup>* (what is the first argument to *ι*?)  
P: *0<sub>ι</sub><sup>1</sup>* (the argument is 0)  
O: *done<sub>ι</sub>* (*ι* terminates)  
P: *done* (whole command terminates).

*Remarks* In the above interaction, at the third move, O was not compelled to ask for the argument to *ι*: if O represented a non-strict procedure, the move *done<sub>ι</sub>* would be played immediately. Similarly, at the fifth move, O could repeat the question *q<sub>ι</sub>* to represent a procedure which calls its argument more than once.

**Strategies.** Using the above ideas, each possible execution of a program is represented as a sequence of moves in the appropriate game. A *program* can therefore be represented as a *strategy* for P, that is, a predetermined way of responding to the moves O makes. A strategy can also choose to make no response in a particular situation, representing divergence, so for example there are two strategies for the game corresponding to **comm**: the strategy for **skip** responds to *run* with *done*, and the strategy for  $\Omega$  fails to respond to *run* at all.

Strategies are usually represented as *sets of sequences of moves*, so that a strategy is identified with the collection of possible traces that can arise if P plays according to that strategy. The fact that O can repeat questions, as we remarked above, means that these sets are very often infinite, even for simple programs. The strategy for the program *ι*(0), for example, is capable of supplying the argument 0 to *ι* as often as O asks for it.

**Interpretation of Variables.** The type **var**[ $\tau$ ] is represented as a game in the following way. For each element *x* of  $\tau$  there is an initial move *write*(*x*), representing an assignment. There is one possible response to this move, *ok*, which signals successful completion of the assignment. For dereferencing, there is an initial move *read*, to which P may respond with any element of *x*.

Here is an interaction in the strategy for

$v : \mathbf{var}[\mathbf{int}] \vdash v := !v + 1.$

O: *run*  
P: *read<sub>v</sub>* (get the value from *v*)  
O: 3 (O supplies the value 3)  
P: *write*(4)<sub>*v*</sub> (write 4 into *v*)  
O: *ok<sub>v</sub>* (the assignment is complete)  
P: *done* (the whole command is complete)

In these interactions, O is *not* constrained to represent a *good variable*, exhibiting the obvious causal dependency between reads and writes. For example, in the game for terms of the form

$\iota : \mathbf{comm}, v : \mathbf{var}[\mathbf{int}] \vdash M : \mathbf{comm}$

we find interactions such as

$run \cdot read_v \cdot 3_v \cdot write(4)_v \cdot ok_v \cdot run_\iota \cdot done_\iota \cdot read_v \cdot 7_v \dots$

Here  $O$  has not played as a good variable; but this freedom is necessary, because our semantics must take care of the case in which  $\iota$  is bound to a procedure which also uses  $v$ , for example, the procedure  $v := 7$ .

There is one situation in which this kind of interference cannot happen: when the variable  $v$  is made local. This has two effects:

- the local interaction with  $v$  is guaranteed to exhibit “good variable” behaviour, and
- the interaction with  $v$  is not an observable part of the programs behaviour.

Therefore, the games interpretation of **new**  $v$  **in**  $M$  is given by taking the set of sequences interpreting  $M$ , considering only those in which  $O$  plays as a good variable in  $v$ , and deleting all the moves pertaining to  $v$ , to hide  $v$  from the outside.

**Full abstraction.** In [1], it was shown that these ideas give rise to a fully abstract model of Idealized Algol, in the following sense. Say that an interaction is *complete* if and only if it begins with an initial move and ends with a move which answers that initial move. Thus, for example,  $run \cdot run_\iota$  is not complete but  $run \cdot run_\iota \cdot done_\iota \cdot done$  is. Then we have the following theorem:

**Theorem 1 (Full Abstraction for IA).** *For any  $\Gamma \vdash P, Q : \theta$ , programs  $P$  and  $Q$  are contextually equivalent in IA ( $P \equiv Q$ ) if and only if the sets of complete plays in the strategies interpreting  $P$  and  $Q$  are equal.*

**Note.** In the above account, a very simple notion of game has been used. In fact, games models require a great deal more machinery, including the notions of *justification pointer* and *questions and answers*, in order for full abstraction to be achieved. The key observation which makes the present paper possible is that, for the interpretation of Idealized Algol up to second-order types, this extra machinery is redundant; it only comes into play at third-order and above.

## 4 Regular Language Game Semantics

We will now give a simple presentation of the game semantics of our fragment of IA. The key idea is that the set of complete plays in a strategy forms a regular language, which leads to a compact notation for defining and manipulating these infinite sets of sequences. We define a metalanguage based on regular expressions, extended with several handy operations: intersection, hiding and relabelling. Of course, these extensions do not change the regular nature of the languages being defined.

**Definition 1.** *The set  $\mathcal{ER}_A$  of extended regular expressions over a finite alphabet  $A$  is defined inductively as the smallest set for which:*

*Constants:*  $\perp, \epsilon \in \mathcal{ER}_{\mathcal{A}}$ ; if  $a \in \mathcal{A}$ , then  $a \in \mathcal{ER}_{\mathcal{A}}$ ;  
*Iteration:* if  $R \in \mathcal{ER}_{\mathcal{A}}$ ,  $R^* \in \mathcal{ER}_{\mathcal{A}}$ ;  
*Operators:* if  $R, S \in \mathcal{ER}_{\mathcal{A}}$ , then  $R \cdot S, R + S, R \cap S \in \mathcal{ER}_{\mathcal{A}}$ ;  
*Hiding:* if  $R \in \mathcal{ER}_{\mathcal{A}}$ ,  $\mathcal{A}' \subseteq \mathcal{A}$ , then  $R|_{\mathcal{A}'} \in \mathcal{ER}_{\mathcal{A}}$ ;  
*Relabelling:* if  $R, S \in \mathcal{ER}_{\mathcal{A}}$ , and  $a, a' \in \mathcal{A}$ , then  $R[a'/a] \in \mathcal{ER}_{\mathcal{A}}$ .

The constant  $\perp$  denotes the empty language, while  $\epsilon$  is the language consisting only of the empty string. The constant  $a$  is the language consisting of the singleton sequence  $a$ .

Hiding represents the operation of restricting a language to a subset  $\mathcal{A} \setminus \mathcal{A}'$  of the original alphabet  $\mathcal{A}$ : the language  $\mathcal{L}(R|_{\mathcal{A}'})$  is the set of sequences in  $\mathcal{L}(R)$ , with all elements of  $\mathcal{A}'$  deleted.

The relabelling operation similarly takes the set of sequences in  $\mathcal{L}(R)$  and replaces each occurrence of  $a$  with  $a'$  to arrive at  $\mathcal{L}(R[a'/a])$ .

The other operations (iteration, concatenation, union, intersection) are defined as usual.

**Proposition 1.** *Every extended regular expression denotes a regular language.*

We now give an extended regular expression representation of the game semantics of our language. We first define the alphabets of interest.

We associate with every type in IA an alphabet, which represents a semantic “domain” over which extended regular expressions will be constructed:

$$\begin{aligned}
 \mathcal{A}[\mathbf{int}] &= \mathcal{N}, & \mathcal{A}[\mathbf{bool}] &= \{tt, ff\} \\
 \mathcal{A}[\mathbf{comm}] &= \{run, done\}, \\
 \mathcal{A}[\mathbf{exp}[\tau]] &= \{q, v \mid v \in \mathcal{A}[\tau]\}, \\
 \mathcal{A}[\mathbf{var}[\tau]] &= \{read, v, write(v), ok \mid v \in \mathcal{A}[\tau]\}, \\
 \mathcal{A}[\sigma_1 \rightarrow \sigma_2 \rightarrow \dots \rightarrow \sigma_k \rightarrow \sigma] &= \{a^k \mid a \in \mathcal{A}[\sigma_k]\} \cup \mathcal{A}[\sigma].
 \end{aligned}$$

By  $a^k$  we mean a lexical operation: the creation of a new symbol by tagging the symbol  $a$  with the numeral  $k$ .

For a term of the form

$$\iota_1 : \theta_1, \iota_2 : \theta_2, \dots, \iota_k : \theta_k \vdash M : \sigma$$

define the *context alphabet* to be the set

$$\bigcup_j \{a_{\iota_j} \mid a \in \mathcal{A}[\theta_j]\}$$

that is, the union of the  $\mathcal{A}[\theta_j]$ s, tagged with the appropriate identifier names as subscripts.

The semantics of a term  $M$  as above is then a regular language of a certain form, defined as follows.

- If  $\sigma = \mathbf{comm}$ , the form is  $run \cdot R_M \cdot done$ .
- If  $\sigma = \mathbf{exp}[\tau]$ , the form is  $\sum_{v \in \mathcal{A}[\tau]} q \cdot R_M^v \cdot v$

$\llbracket \mathbf{skip} \rrbracket = run \cdot done$ $\llbracket n \rrbracket = q \cdot n$ $\llbracket \mathbf{true} \rrbracket = q \cdot tt$ $\llbracket \mathbf{false} \rrbracket = q \cdot ff$ $\llbracket \Omega \rrbracket = \perp$	
$\llbracket l \rrbracket_{\mathbf{com}} = run \cdot run_l \cdot done_l \cdot done$	$\llbracket l \rrbracket_{\mathbf{exp}[\tau]} = \sum_{v \in \mathcal{A}[\tau]} q \cdot q_l \cdot v_l \cdot v$
$\llbracket l \rrbracket_{\mathbf{var}[\tau]} = \sum_{v \in \mathcal{A}[\tau]} (read \cdot read_l \cdot v_l \cdot v + write(v) \cdot write_l(v) \cdot ok_l \cdot ok)$	
$\llbracket \mathbf{while} B \mathbf{do} C \rrbracket = run \cdot (R_B^{tt} \cdot R_C)^* \cdot R_B^{ff} \cdot done$	
$\llbracket V := E \rrbracket = \sum_{v \in \mathcal{A}[\tau]} run \cdot R_E^v \cdot S_V^v \cdot done$	
$\llbracket E_1 = E_2 \rrbracket = \sum_{n \in \mathcal{N}} (q \cdot R_{E_1}^n \cdot R_{E_2}^n \cdot tt) + \sum_{n_1 \neq n_2 \in \mathcal{N}} (q \cdot R_{E_1}^{n_1} \cdot R_{E_2}^{n_2} \cdot ff)$	
$\llbracket E_1 + E_2 \rrbracket = \sum_{n_i \in \mathcal{N}} q \cdot R_{E_1}^{n_1} \cdot R_{E_2}^{n_2} \cdot (n_1 + n_2)$	$\llbracket !V \rrbracket = \sum_{v \in \mathcal{A}[\tau]} q \cdot R_V^v \cdot v$
$\llbracket C; C' \rrbracket = run \cdot R_C \cdot R_{C'} \cdot done$	$\llbracket C; E \rrbracket = \sum_{v \in \mathcal{A}[\tau]} q \cdot R_C \cdot R_E^v \cdot v$
$\llbracket \mathbf{if} B \mathbf{then} C \mathbf{else} C' \rrbracket = run \cdot R_B^{tt} \cdot R_C \cdot done + run \cdot R_B^{ff} \cdot R_{C'} \cdot done$	
$\llbracket \mathbf{if} B \mathbf{then} E \mathbf{else} E' \rrbracket = \sum_{v \in \mathcal{A}[\tau]} (q \cdot R_B^{tt} \cdot R_E^v \cdot v) + \sum_{v \in \mathcal{A}[\tau]} (q \cdot R_B^{ff} \cdot R_{E'}^v \cdot v)$	

**Table 1.** Some semantic valuations

– If  $\sigma = \mathbf{var}[\tau]$ , the form is

$$\sum_{v \in \mathcal{A}[\tau]} (read \cdot R_M^v \cdot v) + \sum_{v \in \mathcal{A}[\tau]} (write(v) \cdot S_M^v \cdot ok)$$

where  $R_M$ ,  $R_M^v$  and  $S_M^v$  are extended regular expressions over the context alphabet of the term  $M$ .

The idea is that, for  $M$  of type **comm**, for example, the language  $R_M$  is the set of interactions with the environment that need to take place for  $M$  to terminate. Similarly,  $R_M^3$  is the set of interactions that an expression  $M$  must have with the environment to return a value of 3, and so on. For  $M$  of type **var** $[\tau]$ ,  $R_M^v$  denotes the interactions required for a value  $v$  to be read from  $M$ , and  $S_M^v$  denotes the interactions needed to write  $v$  into  $M$ .

The semantic valuations for most of the language are given in Table 1, using the  $R_M$  and  $S_M$  notations introduced above.

For example, a trace of  $V := E$  consists of *run* and *done* surrounding the “side-effects” of the assignment: first  $R_E^v$  which is the expression denoting the interaction which leads the expression  $E$  to return value  $v$ , and then  $S_V^v$  which

is the expression denoting the interaction required to write value  $v$  into variable  $V$ .

A trace of a while-loop has the form: some number of repetitions of a trace of the guard which produces “true”, followed by a complete trace of the loop body; and finally, a single trace of the guard producing “false”. Using our semantics, we can easily demonstrate the validity of a typical while-loop equivalence:

$$\begin{aligned}
\llbracket \text{while true do } C \rrbracket & \\
&= \text{run} \cdot (R_{\text{true}}^{tt} \cdot R_C)^* \cdot R_{\text{true}}^{ff} \cdot \text{done} \\
&= \text{run} \cdot (R_C)^* \cdot \perp \cdot \text{done} \\
&= \perp = \llbracket \Omega \rrbracket, \\
&\text{because } R_{\text{true}} = \llbracket \text{true} \rrbracket = q \cdot tt = q \cdot \epsilon \cdot tt + q \cdot \perp \cdot ff \\
&\text{so } R_{\text{true}}^{tt} = \epsilon \text{ and } R_{\text{true}}^{ff} = \perp.
\end{aligned}$$

The semantics of application and of local variables have been omitted from Table 1 because they deserve additional explanation.

**Application.** Let  $\iota$  be a free variable of type  $\sigma_1 \rightarrow \sigma_2 \rightarrow \dots \rightarrow \sigma_k \rightarrow \mathbf{comm}$ , and  $M_1, \dots, M_k$  be terms of type  $\sigma_1, \dots, \sigma_k$ . We then define

$$\llbracket \iota M_1 \dots M_k \rrbracket = \text{run} \cdot \text{run}_\iota \cdot \left( \sum_{j=1}^k \rho_\iota^j \llbracket M_j \rrbracket \right)^* \cdot \text{done}_\iota \cdot \text{done}$$

where  $\rho_\iota^j$  is a relabelling operation that tags the initial and final moves of the arguments  $M_j$  with the identifier which is calling them:

$$\rho_\iota^j(R) = R[w_\iota^j/w], \text{ for } w \in \{\text{run}, \text{done}, q, v, \text{read}, \text{write}(v), \text{ok} \mid v \in \mathcal{A}[\llbracket \tau \rrbracket]\}.$$

**Local variables.** For the semantics of a local variable block, as in the original game semantics, there are two things to do: restrict O’s behaviour to that of a good variable, and hide the interaction with the local variable.

The expression  $\gamma_\tau^\iota$  stipulates that the moves corresponding to  $\iota$  have good-variable behaviour. First, let  $\mathcal{A}[\llbracket \tau \rrbracket]_\iota$  be that part of the alphabet which concerns the variable  $\iota$ :  $\mathbf{var}[\llbracket \tau \rrbracket]$ , that is,

$$\mathcal{A}[\llbracket \tau \rrbracket]_\iota = \{\text{read}_\iota, v_\iota, \text{write}_\iota(v), \text{ok}_\iota \mid v \in \mathcal{A}[\llbracket \tau \rrbracket]\}.$$

Let  $B_\iota$  be the regular language containing all strings which do not contain any elements of  $\mathcal{A}[\llbracket \tau \rrbracket]_\iota$ . If we assume that variables initially hold some default value  $a^\tau$ , then good-variable behaviour is stipulated as follows.

$$\gamma_\tau^\iota = B_\iota \cdot (\text{read}_\iota \cdot a_\iota^\tau \cdot B_\iota)^* \cdot \left( B_\iota \cdot \sum_{v \in \mathcal{A}[\llbracket \tau \rrbracket]} (\text{write}_\iota(v) \cdot \text{ok}_\iota \cdot B_\iota \cdot (\text{read}_\iota \cdot v_\iota \cdot B_\iota)^*) \right)^*$$

For completeness' sake, define  $a^{\text{int}} = 0$  and  $a^{\text{bool}} = \text{ff}$ . We can then give the semantics of blocks as

$$\llbracket \text{new}[\tau] \iota \text{ in } M \rrbracket = (\gamma_\tau^\iota \cap \llbracket M \rrbracket) \upharpoonright_{\mathcal{A}[\llbracket \tau \rrbracket]_\iota}.$$

**Theorem 2. Full abstraction.** *Two terms of the recursion free second order finitary fragment of IA are equivalent (in full IA) if and only if the languages denoted by them are equal:*

$$\text{For any } \Gamma \vdash P, Q : \theta, \quad P \equiv Q \iff \llbracket P \rrbracket = \llbracket Q \rrbracket.$$

*Proof.* We can show that the regular language denoted by a term of IA is equal to the set of complete plays in the fully abstract games semantics [1], therefore the full abstraction property is preserved. Note that language equivalence is asserted outside the fragment we describe here; witnesses to some inequivalences may belong to IA but not to the presented fragment.  $\square$

## 5 Examples of Reasoning

At this point a critical reader may entertain doubts as to our earlier claim of simplicity. We have set up a formal notation for extended regular expressions which includes rather complicated operations. However, the complications are notational and not conceptual. Also, all the operations involved are defined effectively so carrying them out is a mechanical process. We hope that the simplicity of our approach will become clearer when we show examples of reasoning about putative equivalences.

**Locality.** This most simple of equivalences invalidates models of imperative computation relying on a global store model, traceable back to Scott and Strachey [18]. It says that a globally defined procedure can not modify a local variable, and it was first proved in the “possible worlds” model of Reynolds and Oles, constructed using functor categories [11, 12].

$$P : \text{comm} \vdash \text{new } x \text{ in } P \equiv P$$

*Proof.*

$$\begin{aligned} & \llbracket \text{new } x \text{ in } P \rrbracket \\ &= (\gamma^x \cap \llbracket P \rrbracket) \upharpoonright_{\mathcal{A}_x} \\ &= (\gamma^x \cap \text{run} \cdot \text{run}_P \cdot \text{done}_P \cdot \text{done}) \upharpoonright_{\mathcal{A}_x} \\ &= (\text{run} \cdot \text{run}_P \cdot \text{done}_P \cdot \text{done}) \upharpoonright_{\mathcal{A}_x} \\ &\quad \text{because the only possible match is on the first } B_x \text{ of } \gamma_\tau^x \\ &= \text{run} \cdot \text{run}_P \cdot \text{done}_P \cdot \text{done} \\ &= \llbracket P \rrbracket \end{aligned}$$

**Snapback.** This example captures the intuition that changes to the state are in some way irreversible. A procedure executing an argument which is a command

inflicts upon the state changes that can not be undone from within the procedure. This is why in the following, if procedure  $P$  uses its argument both sides will fail to terminate; if procedure  $P$  doesn't use its argument the behaviour of each side will be identical because of the locality of  $x$ , as seen above. The first model to correctly address this issue was O'Hearn and Reynolds's interpretation of IA using the polymorphic linear lambda calculus [7]. Reddy also addressed this issue using a novel "object semantics" approach [14], but in a particular flavour of IA known as interference-controlled ALGOL [5]. A model, building on the previous, that also satisfies this equivalence is O'Hearn and Reddy's [6], a model fully abstract for the second order subset.

$$P : \mathbf{comm} \rightarrow \mathbf{comm} \vdash \\ \mathbf{new } x \mathbf{ in } P(x := 1); \mathbf{ if } !x = 1 \mathbf{ then } \Omega \mathbf{ else skip} \equiv P(\Omega)$$

*Proof.*

$$\begin{aligned} \llbracket x := 1 \rrbracket &= \mathit{run} \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{done} \\ \llbracket P(x := 1) \rrbracket &= \mathit{run} \cdot \mathit{run}_P \cdot (\mathit{run}_P^1 \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{done}_P^1)^* \cdot \mathit{done}_P \cdot \mathit{done} \\ \llbracket \mathbf{ if } !x = 1 \mathbf{ then } \Omega \mathbf{ else skip} \rrbracket &= \sum_{n \neq 1} \mathit{run} \cdot \mathit{read}_x \cdot n_x \cdot \mathit{done} \\ \llbracket P(x := 1); \mathbf{ if } !x = 1 \mathbf{ then } \Omega \mathbf{ else skip} \rrbracket & \\ &= \mathit{run} \cdot \mathit{run}_P \cdot (\mathit{run}_P^1 \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{done}_P^1)^* \cdot \mathit{done}_P \cdot \left( \sum_{n \neq 1} \mathit{read}_x \cdot n_x \right) \cdot \mathit{done} \\ \gamma^x \cap \llbracket P(x := 1); \mathbf{ if } !x = 1 \mathbf{ then } \Omega \mathbf{ else skip} \rrbracket & \\ &= \mathit{run} \cdot \mathit{run}_P \cdot \mathit{done}_P \cdot \mathit{read}_x \cdot 0_x \cdot \mathit{done}, \end{aligned}$$

because the only possibility to complete a trace in  $\sum_{n \neq 1} \mathit{read}_x \cdot n_x$  is if the trace in  $(\mathit{run}_P^1 \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{done}_P^1)^*$  is the empty trace. Otherwise, the good variable property of  $x$  requires  $n_x = 1_x$ , which is banned by the set to which  $n$  is restricted ( $n \neq 1$ ). The meaning of the left hand side is therefore:

$$\begin{aligned} (\gamma^x \cap \llbracket P(x := 1); \mathbf{ if } !x = 1 \mathbf{ then } \Omega \mathbf{ else skip} \rrbracket) \upharpoonright_{\mathcal{A}_x} & \\ &= \mathit{run} \cdot \mathit{run}_P \cdot \mathit{done}_P \cdot \mathit{done} = \llbracket P(\Omega) \rrbracket \end{aligned}$$

**Parametricity.** The intuition of parametricity is one of representation independence. Procedures passed different but equivalent implementations of a same data structure or algorithm are not supposed to be able to distinguish between them. Several such motivating examples are given by O'Hearn and Tennent [8], who introduce a model constructed using a certain relation-preserving functor category.

The specific example we give is of an equivalence of two implementations of a toggle-switch: one which uses 1 for "on" and  $-1$  for "off", and one which uses **true** and **false**. The semantic equations for negation and the inequality test

have not been spelled out but are the obvious ones.

$$\begin{aligned}
P &: \mathbf{comm} \rightarrow \mathbf{exp}[\mathbf{bool}] \rightarrow \mathbf{comm} \vdash \\
\mathbf{new}[\mathbf{int}] \ x \ \mathbf{in} \ x := 1; P(x := -!x)(!x > 0) \\
&\equiv \mathbf{new}[\mathbf{bool}] \ x \ \mathbf{in} \ x := \mathbf{true}; P(x := \mathbf{not} \ x)(!x)
\end{aligned}$$

*Proof.*

$$\begin{aligned}
\llbracket x := -!x \rrbracket &= \sum_{n \in \mathcal{N}} \mathit{run} \cdot \mathit{read}_x \cdot n_x \cdot \mathit{write}_x(-n) \cdot \mathit{ok}_x \cdot \mathit{done} \\
\llbracket !x > 0 \rrbracket &= \sum_{n > 0} q \cdot \mathit{read}_x \cdot n_x \cdot \mathit{tt} + \sum_{n \leq 0} q \cdot \mathit{read}_x \cdot n_x \cdot \mathit{ff} \\
\llbracket x := 1; P(x := -!x)(!x > 0) \rrbracket &= \mathit{run} \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \\
&\quad \mathit{run}_P \cdot \left( \sum_{n \in \mathcal{N}} \mathit{run}_P^1 \cdot \mathit{read}_x \cdot n_x \cdot \mathit{write}_x(-n) \cdot \mathit{ok}_x \cdot \mathit{done}_P^1 + \right. \\
&\quad \left. \sum_{n > 0} q_P^2 \cdot \mathit{read}_x \cdot n_x \cdot \mathit{tt}_P^2 + \sum_{n \leq 0} q_P^2 \cdot \mathit{read}_x \cdot n_x \cdot \mathit{ff}_P^2 \right) \cdot \mathit{done}_P \cdot \mathit{done} \\
\gamma^x_{\mathbf{int}} \cap \llbracket x := 1; P(x := -!x)(!x > 0) \rrbracket &= \\
&= \mathit{run} \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{run}_P \cdot (A + A \cdot B + A \cdot B \cdot A + \dots) \cdot \mathit{done}_P \cdot \mathit{done} \\
&= \mathit{run} \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{run}_P \cdot (A + (A \cdot B)^* \cdot (A + \epsilon)) \cdot \mathit{done}_P \cdot \mathit{done} \\
&\text{where } A = \mathit{run}_P^1 \cdot \mathit{read}_x \cdot 1_x \cdot \mathit{write}_x(-1) \cdot \mathit{ok}_x \cdot \mathit{done}_P^1 \cdot (q_P^2 \cdot \mathit{read}_x \cdot (-1)_x \cdot \mathit{ff}_P^2)^* \\
&\text{and } B = \mathit{run}_P^1 \cdot \mathit{read}_x \cdot (-1)_x \cdot \mathit{write}_x(1) \cdot \mathit{ok}_x \cdot \mathit{done}_P^1 \cdot (q_P^2 \cdot \mathit{read}_x \cdot (1)_x \cdot \mathit{tt}_P^2)^*
\end{aligned}$$

Why this is the case should be intuitively clear. A value of 1 is written into  $x$ , followed by negation only, which constrains all the plays to  $(+1)_x$  and  $(-1)_x$  only. The reads and writes have to match with the good variable behaviour. A fully formal proof is lengthy but trivial, and also possible to handle mechanically. Restricting with  $\lfloor_{\mathcal{A}} \llbracket \mathbf{int} \rrbracket_x$  gives the following trace for the left hand side:

$$\begin{aligned}
&\mathit{run} \cdot \mathit{run}_P \cdot (A + (A \cdot B)^* \cdot (A + \epsilon)) \cdot \mathit{done}_P \cdot \mathit{done} \\
&\text{where } A = \mathit{run}_P^1 \cdot \mathit{done}_P^1 \cdot (q_P^2 \cdot \mathit{ff}_P^2)^* \text{ and } B = \mathit{run}_P^1 \cdot \mathit{done}_P^1 \cdot (q_P^2 \cdot \mathit{tt}_P^2)^*
\end{aligned}$$

A similar calculation on the right hand side leads to the the same result.

## 6 Decidability and Complexity Issues

As we have seen, regular languages provide a semantics for the fragment of IA described here. To manipulate regular languages we have introduced a formal language of extended regular expression, which preserves regularity of the language. All the operations we have used in formulating the semantic valuations have been effectively given. Therefore, we can formulate the following obvious result:

**Theorem 3 (Decidability).** *Equivalence of two terms of the recursion free second order finitary fragment of IA is decidable.*

For the general problem of term equivalence the complexity bound appears to be at least of exponential space, as it is the case for regular expressions with intersection [20]. However, the complexity bound for the general problem may not be relevant for the kind of terms that arise in the model of IA, and particularly for those that would be checked for equivalence in practice. This point, which will be investigated in future work, is of the utmost importance if a tool is to be developed based on our ideas.

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