Amalgams and the Coset Graph

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Chapter 1

Introduction

The aim of this dissertation is to investigate the amalgam method and its uses in modern group theory. First, some historical background will be provided, followed by the main aims of this dissertation.

1.1 Motivation

Group Theory was changed forever with the completion of the proof of the classification of finite simple groups in 1981. After over 10,000 pages of proof it was shown that any finite simple group is isomorphic to one of the following [5, p. 60]:

- C_p – A *cyclic* group of prime order;
- A_n – An *alternating* group of degree $n \geq 5$;
- A finite simple group of *Lie type* (classical, twisted and exceptional); and
- One of the 26 *sporadic* groups.

Since then, much work has gone into understanding and simplifying the proof and also into methods of identifying some of these groups. Two concepts that have been used are amalgams and the coset graph.

An amalgam is a collection of three groups with injections from one of the groups into the other two. The coset graph is a way of describing how the cosets of a group interact with respect to three subgroups. Both of these structures, which can be used to identify special properties of the group, will be discussed in detail: an extensive example will be given and important results proven.

The idea behind the amalgam method is to use graph-theoretic results about the coset graph to infer results about the structure of the amalgam and classify the possible amalgams satisfying certain conditions.

The amalgam method was introduced by Goldschmidt in his paper *Automorphisms of trivalent graphs* [3] where he classified all finite primitive amalgams of index $(3,3)$ in the following theorem:

Theorem 1.1 ([3, Theorem A]). *If G, P_1, P_2 are groups such that:*

- *G is finite and $G = \langle P_1, P_2 \rangle$,*
- *$|P_i : P_1 \cap P_2| = 3$ for $i = 1, 2$, and*
- *no non-trivial normal subgroup of G is contained in $P_1 \cap P_2$.*

Then the pair (P_1, P_2) is isomorphic to exactly one of the following fifteen pairs of groups:

Amalgam	(P_1, P_2)
G_1	(Z_3, Z_3)
G_1^1	(Σ_3, Σ_3)
G_1^2	(Σ_3, Σ_6)
G_1^3	(D_{12}, D_{12})
G_2	(D_{12}, A_4)
G_2^1	(D_{24}, Σ_4)
G_2^2	$(D_8\lambda\Sigma_3, \Sigma_4)$
G_2^3	$(D_{12} \times Z_2, A_4 \times Z_2)$
G_2^4	$(D_8 \times \Sigma_3, Z_2 \times \Sigma_4)$
G_3	(Σ_4, Σ_4)
G_3^1	$(Z_2 \times \Sigma_4, Z_2 \times \Sigma_4)$
G_4	$((Q_8 \rtimes Z_4)\Sigma_3, (Z_4 \times Z_4)\Sigma_3)$
G_4^1	$((Q_8 \rtimes Q_8)^1\Sigma_3, (Z_4 \times Z_4)D_{12})$
G_5	$((Q_8 \rtimes Q_8)^2\Sigma_3, (Z_4 \times Z_4)D_{12})$
G_5^1	$((Q_8 \rtimes Q_8)^2 D_{12}, (Z_4 \times Z_4)\Sigma_3\lambda D_8)$

Where $(Q_8 \rtimes Q_8)^n\Sigma_3$ is a semi-direct product with n non-central 2-chief factors ($n = 1, 2$) and $(Q_8 \rtimes Q_8)^n D_{12}$ is a non-split extension.

Since then, there have been many papers about identifying amalgams using the amalgam method. In [2] Delgado and Stellmacher classified P_1 and P_2 for rank 2 amalgams where

- $G = \langle P_1, P_2 \rangle$,
- B is the normalizer of a Sylow p-subgroup in P_i,
- No nontrivial normal subgroup of B is normal in G, and

- $O^{p'}(P_i/O_p(P_i))$ is a rank 1 Lie-type group in char p (including soluble ones).

This work was then continued in [12] where Stellmacher and Timmesfeld partially classified P_1, P_2 and P_3 for rank 3 amalgams where

$$G = \langle P_1, P_2, P_3 \rangle$$

and the other 3 hypotheses above are also satisfied.

1.2 Prerequisites

It will be assumed that the reader has a background knowledge of group theory (a level appropriate to a third year undergraduate student). For those readers without this knowledge of group theory it is recommended that they read *Topics in Group Theory*, by Smith and Tabachnikova [11].

1.3 Aims

A treatment of the general amalgam method is beyond the scope of this dissertation, however, the aim will be to demonstrate the method in action by working through an example with constraints to simplify proceedings.

First some preliminary group theory will be given, before the definitions of an amalgam and a coset graph are given along with some key properties. Then some key results, interesting in their own right, will be proven about these objects. The group-theoretic properties of the amalgam and graph-theoretic properties of the coset graph will become inextricably linked through some pivotal theorems.

We are then in a position to work through an example of the amalgam method, taken from [7]. There will be comparisons to the general amalgam method as well as two concrete examples demonstrating the properties proven.

Finally, there will be a brief overview of how amalgams and the coset graph generalize. This includes amalgams based on n subgroups, defining amalgams, the coset geometry and coset complex. One of the most interesting results is that the link between amalgams and geometry extends to the simplicial coset complex where group-theoretic properties are related to topological properties of the simplicial complex.

Chapter 2

Preliminaries and Definitions

We first recall some basic group theory definitions, then give definitions of an amalgam, coset graph and their properties. Definitions in this section are from [1] and [7].

2.1 Basic Group Theory

Definition 2.1. For a group G we let $G^{\#}$ denote the set

$$G^{\#} := \{g \in G \mid g \neq e\}.$$

Definition 2.2. For a group G, a subgroup H is **maximal** if whenever $U \leq G$ such that $H \leq U$ then $U = G$. That is, H is not contained in a proper subgroup of G.

Definition 2.3. Let G be a group. Define the **Frattini subgroup**, $\Phi(G)$, to be the intersection of all maximal subgroups of G.

Definition 2.4. We say that an Abelian p-group, G, is **elementary Abelian** if $x^p = 1$ for all $x \in G$.

Lemma 2.5 ([7, 2.1.8])**.** *Let G be an elementary Abelian p-group and let the order of G be p^n. Then*

$$G \cong \underbrace{C_p \times C_p \times \cdots \times C_p}_{n \text{ copies of } C_p} \cong (C_p)^n.$$

Proof. This follows immediately from the classification of finitely generated abelian groups (see [11] page 81).

□

Corollary 2.6 ([7, 5.2.6])**.** *Let G be an elementary Abelian p-group. Then its Frattini subgroup is trivial, that is*

$$\Phi(G) = 1.$$

Proof. This follows from Lemma 2.5. □

Lemma 2.7 ([7, 5.2.7])**.** *Let P be a p-group. Then*

(a) $P/\Phi(P)$ is elementary Abelian.

(b) If $|P/\Phi(P)| = p^n$, then there exist $x_1, x_2, \ldots, x_n \in P$ so that

$$P = \langle x_1, x_2, \ldots, x_n \rangle.$$

Proof. (a) In a nilpotent group every maximal subgroup is normal of index p. The result therefore follows from the fact

$$P/\Phi(P) \cong P/P_1 \times P/P_2 \times \cdots \times P/P_m$$

where P_i are maximal subgroups.

(b) We use (a) and Lemma 2.5 to see that $P/\Phi(P)$ is generated by n elements $x_1\Phi(P)$, $x_2\Phi(P)$, $\ldots$, $x_n\Phi(P)$ with all x_i in P. Therefore

$$P = \langle x_1, x_2, \ldots, x_n \rangle \Phi(P).$$

We now use the fact that if a subgroup H of G is such that $G = H\Phi(G)$ then $G = H$, to see that $P = \langle x_1, x_2, \ldots, x_n \rangle$. □

Definition 2.8. Let G be a group. A non-identity element g of G is said to be an **involution** if g has order 2; that is

$$g^2 = e \qquad \text{and} \qquad g \neq e.$$

Definition 2.9. For G a group and $H \leq G$ we define the **normalizer**, $N_G(H)$, of H in G to be

$$N_G(H) := \{g \in G \mid gH = Hg\}.$$

So by definition $H \trianglelefteq N_G(H)$ and $N_G(H)$ is the largest subgroup of G having H as a normal subgroup. Note also, that if $H \trianglelefteq G$ then $N_G(H) = G$.

Definition 2.10. For a group G we define an **automorphism** of G to be an isomorphism $\varphi : G \longrightarrow G$.

We define $\mathrm{Aut}(G)$ to be the group of all automorphisms of G.

Definition 2.11. For a subgroup $H \leq G$ we say H is **characteristic** in G, written H char G, if H is $\mathrm{Aut}(G)$-invariant; that is, for all $\varphi \in \mathrm{Aut}(G)$ and all $h \in H$ we have $\varphi(h) \in H$.

Lemma 2.12 ([7, 1.3.2])**.** *If $H \trianglelefteq G$ and K char H, then $K \trianglelefteq G$.*

Proof. For all $g \in G$, define the automorphism $\psi_g : G \longrightarrow G$ sending x to x^g. As H is a normal subgroup of G, for all $g \in G$ we have $H^g = H$, so $\psi_g|_H$is an automorphism of H for all $g \in G$.

Now K char H, so is invariant under automorphisms of H. Therefore K is invariant under $\psi_g|_H$ for all $g \in G$, so for all $k \in K$ and $g \in G$, $\psi_g|_H(k) = k^g \in K$ and so $K \trianglelefteq G$.

□

Definition 2.13. Let G be a p-group and denote

$$\Omega(G) := \langle x \in G \mid x^p = 1 \rangle.$$

Then it is clear that $\Omega(G)$ is a characteristic subgroup of G and if G is Abelian then

$$G \text{ is elementary Abelian} \iff G = \Omega(G).$$

Definition 2.14. For G a finite group and p a prime we define the **p-radical of** G, $O_p(G)$, to be the largest normal p-subgroup of G. This can be shown to be equal to

$$O_p(G) := \prod_{\substack{A \trianglelefteq G \\ A \in \mathcal{P}}} A$$

where $\mathcal{P}$ is the set of all p-groups.

Definition 2.15. For G a finite group and p a prime we define the **p-residue of** G, $O^p(G)$, to be

$$O^p(G) := \bigcap_{\substack{A \trianglelefteq G \\ G/A \in \mathcal{P}}} A$$

where $\mathcal{P}$ is the set of all p-groups.

Definition 2.16. We define a **linear representation** of a group G on a vector space V over a field F to be a homomorphism

$$\varphi : G \longrightarrow \mathrm{GL}(V).$$

We define the **degree** of φ to be the degree of V over F.

Definition 2.17. We define a **permutation representation** of a group G on a set Ω to be a homomorphism

$$\varphi : G \longrightarrow \mathrm{Sym}(\Omega).$$

We define the **degree** of φ to be $|\Omega|$.

Definition 2.18. For a group G acting (on the right) on a set Ω, we define the **stabilizer** of an element $\omega \in \Omega$, $\mathrm{stab}_G(\omega)$, to be

$$\mathrm{stab}_G(\omega) = G_\omega = \{g \in G \mid \omega \cdot g = \omega\}.$$

Definition 2.19. For a group G acting (on the right) on a set Ω we define a **set of imprimitivity** to be a non-empty, proper subset $\Delta \subset \Omega$ such that for every $g \in G$ either:

$$\Delta \cdot g = \Delta \text{ or } (\Delta \cdot g) \cap \Delta = \emptyset.$$

It is clear that if Δ is a set of imprimitivity, then so is Δ^g.

Definition 2.20. We say the action of the group A on G is **coprime** if

- $(|A|, |G|) = 1$, and
- A or G is soluble.

Note. By Feit–Thompson one of A or G must be soluble if their orders are coprime.

Definition 2.21. Let G be a group. We say that G is of **even type** if the connected component of a Sylow 2-subgroup is unipotent and non-trivial.

2.2 Amalgams

We now give a formal definition of an amalgam and some basic properties associated with amalgams.

Definition 2.22. An **amalgam** is a quintuple of three groups A, B and C and two injective homomorphisms $\varphi_1 : C \longrightarrow A$ and $\varphi_2 : C \longrightarrow B$. We denote the amalgam as $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$.

$$A \xleftarrow{\varphi_1} C \xrightarrow{\varphi_2} B$$

Figure 2.1: Amalgam

Definition 2.23. The **index** of an amalgam $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ is the ordered pair $(|A : \varphi_1(C)|, |B : \varphi_2(C)|)$.

Definition 2.24. We say an amalgam, $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$, is **simple** if for every $1 \neq K \leq C$ one, or both, of the following holds:

$$\begin{aligned} \varphi_1(K) &\ntrianglelefteq A; \\ \varphi_2(K) &\ntrianglelefteq B. \end{aligned}$$

Definition 2.25. A **completion**, denoted (G, ψ_1, ψ_2), of an amalgam $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ is a group G and homomorphisms $\psi_1 : A \longrightarrow G$ and $\psi_2 : B \longrightarrow G$ such that $G = \langle \psi_1(A), \psi_2(B) \rangle$ and $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. This is equivalent to saying that Figure 2.2 commutes.

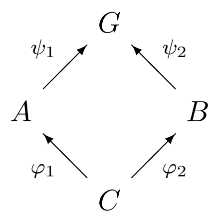

Figure 2.2: Completion of an Amalgam

We say a completion is **faithful** if and only if both ψ_1 and ψ_2 are injective.

Definition 2.26. Let G be a group. The **canonical presentation** is

$$\langle G \mid R(G) \rangle$$

where $R(G)$ is the kernel of the homomorphism from the free group on G, $F(G)$, to G, sending each generator of $F(G)$ to the corresponding element of G.

Definition 2.27. Let $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ be an amalgam. Let $\langle X_1 \mid R_1 \rangle$ and $\langle X_2 \mid R_2 \rangle$ be the canonical presentations of A and B, where $X_1 \cap X_2 = \emptyset$. Then the **amalgamated free product**, denoted $A *_C B$, also known as the **universal completion**, is the group

$$\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\varphi_1(g) = \varphi_2(g) \mid g \in C\} \rangle.$$

Every completion of $\mathcal{A}$ can be obtained by a group homomorphism from the universal completion.

Note. This notation is ambiguous, as the definition of $A *_C B$ depends on φ_1 and φ_2; however, the notation has become standard so we will follow convention.

Definition 2.28. For an amalgam $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ we can identify A, B and C with their images in $A *_C B$, so φ_1 and φ_2 are seen as inclusion maps. We say that $\mathcal{A}$ is **primitive** if:

- For all $1 \neq K \trianglelefteq A$ with $K \subseteq C$, $C = N_B(K)$; and
- For all $1 \neq L \trianglelefteq B$ with $L \subseteq C$, $C = N_A(L)$.

Note. If C is a maximal subgroup of A and B then primitivity is equivalent to A and B having no common normal subgroups.

2.3 The Coset Graph

We now give definitions of the coset graph and some of its graph theoretic properties.

Definition 2.29. Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Then we say Γ is **bipartite** if $V(\Gamma)$ can be partitioned into two sets V_1 and V_2 such that edges in $E(\Gamma)$ only connect vertices in different vertex classes (so there are no edges between two vertices in the same class).

Definition 2.30. The **(right) coset graph**, $\Gamma = \Gamma(G, P_1, P_2, C)$, of subgroups P_1 and P_2 of G with $C \leq P_1 \cap P_2$ is the graph with vertex set

$$V(\Gamma) = \{P_i g \mid g \in G, i = 1, 2\}$$

and edges between vertices representing cosets that share a coset of C in G, so

$$E(\Gamma) = \{Ck \mid g \in G\}.$$

Therefore two vertices $P_1 g$ and $P_2 h$ are joined by the edge Ck if and only if $Ck \subseteq P_1 g \cap P_2 h$. It is clear that Γ is a bipartite graph with disjoint vertex classes $\{P_1 g \mid g \in G\}$ and $\{P_2 g \mid g \in G\}$.

Note. $\Gamma(G, P_1, P_2, C)$ is never the empty graph (that is, the graph with no edges). By setting k, g and h as the identity we see P_1 and P_2 are connected by the edge C since, by definition, $C \leq P_1 \cap P_2$ so certainly $C \subseteq P_1 \cap P_2$.

Definition 2.31. Let Γ be a graph (or subgraph) with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We say that two vertices α and β are **connected** if and only if there is a path of edges in $E(\Gamma)$ from α to β. We say that the graph Γ is **connected** if every pair of vertices α and β in $V(\Gamma)$ are connected.

Definition 2.32. The **distance**, $d(\alpha, \beta)$, of two vertices $\alpha, \beta \in V(\Gamma)$ is defined to be the length of the shortest path between them. We define $d(\alpha, \alpha)$ to be zero for all vertices α; and if α and β are not connected we let $d(\alpha, \beta) = \infty$.

Definition 2.33. For a connected component $\widehat{\Gamma} \subseteq \Gamma$ we let the **diameter of** $\widehat{\Gamma}$ be the least integer n such that the distance between any pair of vertices is at most n.

We also denote for any vertex $\alpha \in V(\Gamma)$, and any $j \in \mathbb{N}$

$$\Delta_j(\alpha) = \{\beta \in \Gamma \mid d(\alpha, \beta) = j\}.$$

For the case $j = 1$ we simplify the notation so that the set of adjacent vertices to α is written simply as

$$\Delta(\alpha) = \{\beta \in \Gamma \mid d(\alpha, \beta) = 1\}.$$

Definition 2.34. For a pair of vertices $\alpha, \beta \in V(\Gamma)$ we define the **edge set**, $E(\alpha, \beta)$, to be the set of all edges between α and β, that is:

$$E(\alpha, \beta) := \{Ck \mid Ck \subseteq \alpha \cap \beta\}.$$

Definition 2.35. For a vertex $\alpha \in V(\Gamma)$ and for i a natural number we denote

$$G_\alpha^{[i]} := \bigcap_{\substack{\delta \in V(\Gamma) \\ d(\alpha,\delta) \leq i}} G_\delta.$$

2.4 Examples

Example 2.36. We give an easy example of an amalgam as follows: let

- $A = S_3$;
- $B = S_4$;
- $C = A_3$; and
- $\varphi_1 : C \hookrightarrow A$ and $\varphi_2 : C \hookrightarrow B$ the obvious inclusion maps.

Then $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ is an amalgam of index $(2, 8)$. We have a completion (G, ψ_1, ψ_2), where $G = S_4$ and $\psi_1 : A \hookrightarrow G$ and $\psi_2 : B \hookrightarrow G$ are the obvious inclusion maps, and it is faithful.

Example 2.37. We will now construct the coset graph of the following quadruple of groups:

- $G = S_4$;
- $P_1 = A_4$;
- $P_2 = V_4$; and
- $C = \langle (1\ 2)(3\ 4) \rangle$.

Now there are only two right cosets of A_4 in S_4, namely A_4 and $A_4(12)$. On the other hand, V_4 has 6 distinct left cosets: V_4, $V_4(12)$, $V_4(23)$, $V_4(13)$, $V_4(123)$ and $V_4(132)$. We know, therefore, that $\Gamma(S_4, A_4, V_4, \langle(12)(34)\rangle)$ is a bipartite graph with vertex classes of size 2 and 6. It is a simple calculation to see that A_4 and $A_4(12)$ both have three neighbouring vertices, each connected by 4 edges. This results in the coset graph shown in Figure 2.3, where $u_1 = A_4$, $u_2 = A_4(12)$ and $v_1 = V_4$, $v_2 = V_4(123)$, $v_3 = V_4(132)$, $v_4 = V_4(12)$, $v_5 = V_4(23)$ and $v_6 = V_4(13)$. The number of edges has been written above each edge for simplicity.

In this case we have, for example:

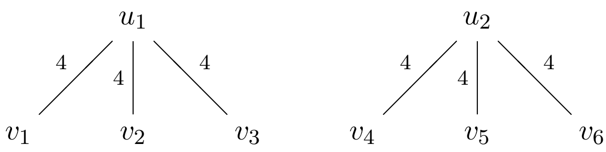

Figure 2.3: Example coset graph

- $d(v_1, v_3) = 2$, $d(u_1, v_1) = 1$ and $d(u_1, u_2) = \infty$;
- $\Delta(u_2) = \{v_4, v_5, v_6\}$; and
- the diameter of each connected component of Γ is 2.

Chapter 3

Important Results Concerning Amalgams and the Coset Graph

We now prove some key results: that every amalgam $\mathcal{A}$ has a faithful completion; that the coset graph Γ is connected if and only if the group is generated by the specified subgroups P_1 and P_2; and various results concerning the stabilizers of Γ and the action of G on Γ.

3.1 Amalgams

One of the key results in amalgam theory is that every amalgam has a faithful completion. We can see that every amalgam has a completion by simply taking G to be the trivial group and letting φ_1, and φ_2, map all elements of A, and B respectively, to the identity element. However, although completions always exist it is not obvious why there should exist a completion that is faithful. We prove this result now, following [1].

Theorem 3.1 ([1, 4.1]). *Given an amalgam $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ there exists a faithful completion. Also, if A and B are finite there exists a finite completion.*

Proof. Let $\mathcal{S}$ be a left transversal of $\varphi_1(C)$ in A and $\mathcal{T}$ a left transversal of $\varphi_2(C)$ in B. Then any element a in A can be written in a unique way as $s\varphi_1(c)$ for some s in $\mathcal{S}$ and some c in C. Similarly, any b in B can be written as $t\varphi_2(c)$ for a unique element t in $\mathcal{T}$ and c in C.

Now denote:

$$\Omega := A \times \mathcal{T} = \{(a, t) \mid a \in A, t \in \mathcal{T}\},$$

and

$$\Omega' := \mathcal{S} \times B = \{(s, b) \mid s \in \mathcal{S}, b \in B\}.$$

We define an action of A on Ω, denoted $\bullet$:

$$(a,t) \bullet a_1 := (aa_1, t),$$

and define a similar action of B on Ω', denoted $\circ$:

$$(s,b) \circ b_1 := (s, bb_1).$$

Now define $\Theta : \Omega \longrightarrow \Omega'$ by

$$\Theta(s\varphi_1(c), t) = (s, t\varphi_2(c)).$$

We aim to show that Θ is a bijection and then construct permutation representations into Ω. First we will see that Θ is onto. Let (s,b) be any element of Ω'. Then there exists t in $\mathcal{T}$ and c in C such that $b = t\varphi_2(c)$ and so $(s,b) = (s, t\varphi_2(c))$. Now $s\varphi_1(c)$ is in A, so $(s\varphi_1(c), t)$ is in Ω and so

$$\Theta : (s\varphi_1(c), t) \mapsto (s, t\varphi_2(c)) = (s,b).$$

Hence Θ is onto.

Now we show that Θ is injective. To see this, suppose $\Theta(a,t) = \Theta(a',t')$. Then there exist c and c' in C and s and s' in $\mathcal{S}$ such that

$$(a,t) = (s\varphi_1(c), t),$$

and

$$(a',t') = (s'\varphi_1(c'), t').$$

Now,

$$\begin{aligned}\Theta(s\varphi_1(c), t) = \Theta(s'\varphi_1(c'), t') &\Leftrightarrow (s, t\varphi_2(c)) = (s', t'\varphi_2(c')) \\ &\Leftrightarrow s = s', t\varphi_2(c) = t'\varphi_2(c').\end{aligned}$$

Since any b in B is written uniquely as $t\varphi_2(c)$ we must have $t = t'$ and $\varphi_2(c) = \varphi_2(c')$. Hence we have $c = c'$, as φ_2 is injective, and then we must have $\varphi_1(c) = \varphi_1(c')$. Hence

$$(a,t) = (s\varphi_1(c), t) = (s'\varphi_1(c'), t') = (a', t'),$$

and so Θ is injective and hence a bijection.

With this bijection we can define the following action of B on Ω, denoted $*$:

$$(a,t) * b = \Theta^{-1}((\Theta(a,t)) \circ b).$$

Notice that given any $d \in C$, $\varphi_1(d)$, as an element of A, acts on Ω by

$$(s\varphi_1(c), t) \bullet \varphi_1(d) = (s\varphi_1(c)\varphi_1(d), t) = (s\varphi_1(cd), t)$$

and $\varphi_2(d)$, as an element of B, acts on Ω by

$$\begin{aligned}(s\varphi_1(c),t) * \varphi_2(d) &= \Theta^{-1}((\Theta(s\varphi_1(c),t)) \circ \varphi_2(d)) \\ &= \Theta^{-1}((s,t\varphi_2(c)) \circ \varphi_2(d)) \\ &= \Theta^{-1}(s,t\varphi_2(c)\varphi_2(d)) \\ &= \Theta^{-1}(s,t\varphi_2(cd)) \\ &= (s\varphi_1(cd),t).\end{aligned}$$

Hence $\varphi_1(c)$ acts on Ω in the same way as $\varphi_2(c)$ acts on Ω. Notice also that both these actions on Ω are faithful and so there exist permutation representations

$\Psi_1 : A \longrightarrow \mathrm{Sym}(\Omega)$, and

$\Psi_2 : B \longrightarrow \mathrm{Sym}(\Omega)$

which are injective.

So $G := \langle \Psi_1(A), \Psi_2(B) \rangle$ is a faithful completion of $\mathcal{A}$. Moreover if A and B are finite then it is clear that G is a finite faithful completion.

□

3.2 The Coset Graph

We now prove some important results about the coset graph to facilitate its application to the amalgam method.

Lemma 3.2. *Let $\Gamma = \Gamma(G, P_1, P_2, C)$ be a coset graph. If $|P_1 \cap P_2 : C|$ is $n \in \mathbb{N}$ then for any pair of adjacent vertices $\{P_i g, P_j h\} \in E(\Gamma)$ there are exactly n edges between $P_i g$ and $P_j h$.*

Proof. As $|P_1 \cap P_2 : C| = n$, there are exactly n cosets of C in $P_1 \cap P_2$. Let

$$\mathcal{C} = \{t_i \mid t_i \in P_1 \cap P_2, i = 1, \ldots, n\}$$

be a left transversal of C in $P_1 \cap P_2$.

As $P_i g$ and $P_j h$ are adjacent there exists k in G such that

$$Ck \subseteq P_i g \cap P_j h.$$

It is also clear that $Ck \subseteq P_i g \cap P_j h$ if and only if $P_i g = P_i k$ and $P_j h = P_j k$ as cosets are disjoint. We therefore only need to consider cosets of C in $P_i k \cap P_j k$. There must be at least n such cosets: $Ct_1k, Ct_2k, \ldots, Ct_nk$.

Now if $Ct \subseteq P_i g \cap P_j h = (P_i \cap P_j)k$, then we have $Ctk^{-1} \subseteq P_i \cap P_j$ so $Ct = Ct_i k$ for some $t_i \in \mathcal{C}$. Hence there cannot be any other cosets and so the number of edges between $P_i g$ and $P_j h$ is exactly n.

□

3.2.1 The Action of a Group on its Coset Graph

The group G acts on the coset graph $\Gamma(G, P_1, P_2, C)$ by acting on the cosets with right multiplication, so for a vertex $\alpha = P_i h$ and $g \in G$

$$\alpha \cdot g := P_i(hg).$$

This action clearly preserves the graph structure: if Ck is an edge of Γ between vertices P_1h_1 and P_2h_2, then for any g in G we automatically have

$$Ckg \subseteq P_1h_1g \cap P_2h_2g,$$

and so the image of the Ck is an edge between the images of the vertices P_1h_1 and P_2h_2.

The action of G on Γ has two orbits, namely the set of cosets of P_1 and the set of cosets of P_2. G also acts transitively on cosets of C and so is transitive on the edges of Γ.

Lemma 3.3 ([7, 10.3.1]). *(a) For $\alpha = P_i h \in V(\Gamma)$, the stabilizer G_α is conjugate to P_i, specifically $G_\alpha = P_i^h$.*

(b) For an edge $e = Ck \in E(\Gamma)$, the stabilizer G_e is conjugate to C.

(c) The kernel of the action of G on Γ is $(P_1 \cap P_2)_G$.

Proof. (a) For a vertex $\alpha = P_i h$ and $g \in G$ acting on α it is clear

$$\alpha \cdot g = P_i hg = P_i h \Leftrightarrow g \in h^{-1}P_i h = P_i^h,$$

and so $G_\alpha = P_i^h$.

(b) Let $e = Ck$ be an edge. Then the stabilizer of the edge e is the set

$$\{g \in G \mid Ck \cdot g = Ck\}$$

which contains those elements such that $kgk^{-1} \in C$. So $G_e = C^k$.

(c) By part (a) any normal subgroup of G contained in $P_1 \cap P_2$ fixes all vertices in Γ.

□

Lemma 3.4 ([1, 4.4]). *For any vertex α, G_α is transitive on $\Delta(\alpha)$ and the sets $E(\alpha, \beta)$ form sets of imprimitivity.*

Proof. Let α and β be adjacent and without loss of generality let $\alpha = P_1 g$ and $\beta = P_2 h$. We saw in Lemma 3.2 that $\alpha = P_1 k$, $\beta = P_2 k$ and

$$E(\alpha, \beta) = \{Ct_1k, Ct_2k, \ldots, Ct_nk\}$$

where the t_i are elements of $P_1 \cap P_2$. By 3.3(a) we know $G_\alpha = P_1^k$. Consider two edges, Ct_ik and Ct_jk. Then the element $t := k^{-1}t_i^{-1}t_jk$ sends Ct_ik to

Ct_jk and t is in G_α. It is clear that in fact t is in $G_{\alpha\beta}$ which is equal to the stabilizer of any edge between α and β, and so is equal to C^k. Therefore $G_{\alpha\beta}$ acts transitively on $E(\alpha, \beta)$.

Now consider β and γ in $\Delta(\alpha)$ with $\beta \neq \gamma$. Let $Ck_1 \subseteq \alpha \cap \beta$ and $Ck_2 \subseteq \alpha \cap \gamma$. Then, from 3.2, we have $\beta = P_2k_1$, $\gamma = P_2k_2$ and

$$\alpha = P_1k_1 = P_1k_2.$$

We therefore have $k_2k_1^{-1}$ in P_1 and so

$$x := k_1^{-1}k_2 = k_1^{-1}k_2k_1^{-1}k_1 \in P_1^{k_1} = G_\alpha.$$

It is clear x takes β to γ and the other cosets between α and β are of the form Ct_ik_1 with t_i in $P_1 \cap P_2$. Hence $Ct_ik_1 \cdot x = Ct_ik_2$, an edge between α and γ. Therefore x takes β to γ, and G_α is transitive on $\Delta(\alpha)$.

We saw before that G_α acts transitively on each $E(\alpha, \beta)$. Now any element of G_α has the form $y := k^{-1}zk$ with $z \in P_1$. Now

$$Ct_ik \cdot y = Ct_izk \in E(\alpha, \gamma)$$

for some $\gamma \in \Delta(\alpha)$. This holds for all $1 \leq i \leq n$ and so $\Delta(\alpha) \cdot g = \Delta(\alpha)$ for all g in G_α and so $\Delta(\alpha)$ is a set of imprimitivity.

□

Lemma 3.5 ([1, 4.7]). *The kernel of the action of G on Γ is the largest normal subgroup of G which is contained in C.*

Proof. Denote by N the largest normal subgroup of G contained in C.

If K is the kernel of the action of G on Γ then every element of K fixes every vertex and edge of Γ. So in particular it fixes the edge C and so $K \leq C$. As $K \trianglelefteq G$ and N is maximal, we have $K \leq N$.

As $N \trianglelefteq G$ for every n in N and g in G we have $n^{g^{-1}} \in N \leq C \leq P_i$ so $P_ign = P_ig$ for $i = 1, 2$ and P_ig is in $V(\Gamma)$, so N fixes all vertices in Γ. It is clear that we also have $Cgn = Cg$ for all edges of Γ, so $N \leq K$. Equality therefore holds.

□

3.2.2 Connectedness of the Coset Graph

We now relate the connectedness of the coset graph to a group-theoretic property; that is, whether G is generated by P_1 and P_2.

Theorem 3.6 ([1, 4.6]). *The coset graph $\Gamma = \Gamma(G, P_1, P_2, C)$ is connected if and only if $G = \langle P_1, P_2 \rangle$.*

Proof. Firstly, assume $G = \langle P_1, P_2 \rangle$. We know that C is an edge between P_1 and P_2. Let Γ_* be the connected component containing the edge C. P_1 and P_2 are both in Γ_* and because disconnected components of Γ are disjoint we find that

$$\Gamma_* = \Gamma_*^{\langle P_1, P_2 \rangle}.$$

Hence as $G = \langle P_1, P_2 \rangle$ we have Γ_* invariant under right conjugation by G. By Lemma 3.3(a) this means that Γ_* is invariant under the stabilizers of all the vertices in $V(\Gamma)$. However, we then see that all the vertices must be in Γ_*, so $\Gamma_* = \Gamma$, and Γ is connected.

Now assume Γ is connected. Let $G_0 = \langle P_1, P_2 \rangle$ and assume $G_0 < G$. Let $\alpha = P_1$ and choose a $\beta = P_i x$ such that β is not in G_0 and $n = d(\alpha, \beta)$ is minimal. We will show that in fact β is in G_0 thus deriving a contradiction.

Γ is connected so there exists a path from α to β. Choose a shortest such path, and denote the vertices γ_i $(i = 0, \ldots, n)$, where $\gamma_0 = \alpha$ and $\gamma_n = \beta$. We have chosen n minimal, so $\gamma_{n-1} = P_i g$ and $\gamma_{n-2} = P_j h$ for some $g, h \in G_0$. By the result in Lemma 3.4 we know $G_{\gamma_{n-1}} = P_i^g$ is transtive on $\Delta(\gamma_{n-2})$ so there exists a $k \in P_i^g \leq G_0$ with $\gamma_{n-2} \cdot k = \beta$. However, then β would be in G_0, the desired contradiction, and so $G = G_0 = \langle P_1, P_2 \rangle$.

□

We now use Theorem 3.6 to prove a result about how subgroups of the vertex stabilizers act on the graph.

Theorem 3.7 ([7, 10.3.3])**.** *Let $G = \langle P_1, P_2 \rangle$. Suppose that $\{\alpha, \beta\}$ is an edge of Γ, $U \leq G_\alpha \cap G_\beta$ and one of the following holds:*

(a) $N_{G_\alpha}(U)$ acts transitively on $\Delta(\delta)$ for $\delta \in \{\alpha, \beta\}$; or

(b) $U \trianglelefteq G_\alpha$ and $U \trianglelefteq G_\beta$.

Then U acts trivially on Γ.

Proof. It is clear that assuming (b) with 3.4 implies (a) and so we can assume that (a) holds. Now let

$$\Gamma_0 := (\alpha)^{N_G(U)} \cup (\beta)^{N_G(U)}.$$

Then U fixes every vertex in Γ_0. Let γ be in Γ, so there exists x in $N_G(U)$ and δ in $\{\alpha, \beta\}$ such that $\gamma = \delta^x$.

Now

$$\Delta(\delta^x) = \Delta(\gamma) \quad \text{and} \quad N_{G_\gamma}(U) = N_{G_\delta}(U)^x,$$

and by (a), $N_{G_\gamma}(U)$ is transitive on $\Delta(\delta^x) = \Delta(\gamma)$. Moreover, one of the vertices in $\{(\alpha)^x, (\beta)^x\}$ is adjacent to γ and

$$\{(\alpha)^x, (\beta)^x\} \subseteq \Gamma_0.$$

It follows that $\Delta(\gamma) \subseteq \Gamma_0$. Since, by Theorem 3.6, Γ is connected we conclude that $\Gamma = \Gamma_0$. Thus U stabilizes every vertex in Γ.

□

Chapter 4

The Amalgam Method

In this chapter we will follow [7, 10.3] to demonstrate the amalgam method in action, and we will compare the example to the general amalgam method given in [4]. We will also see how the amalgam method can be used to help identify groups.

4.1 Motivation

The amalgam method has been used to aid the proof of the classification of finite simple groups [4, pp. 161–169]. It is used by considering Sylow 2-subgroups of G and their normalizers.

Let G be a group of even type (see Definition 2.21) and T a Sylow 2-subgroup of G. When considering N_T, the normalizer of T, we have a dichotomy: N_T is either contained in a unique maximal 2-local subgroup of G, or this subgroup is not unique.

The first case is tackled using a technique called near components which is beyond the scope of this dissertation but is discussed in [4].

In the second case there exists at least two maximal 2-local subgroups and we can form an amalgam from two or more of these M_1, M_2, $\ldots$, M_n, as in Figure 4.1. In this context the amalgam method, with a few additional assumptions, can be applied to assist in the identification of the groups involved.

4.2 Assumptions

In our example we will assume the following conditions to restrict our conclusion to only two cases:

Assumption ($\mathcal{G}$)**.** Let G be a group generated by two finite subgroups P_1 and P_2, and let $T := P_1 \cap P_2$. Suppose for $i = 1, 2$:

$\mathcal{G}_1$ $C_{P_i}(O_2(P_i)) \leq O_2(P_i)$;

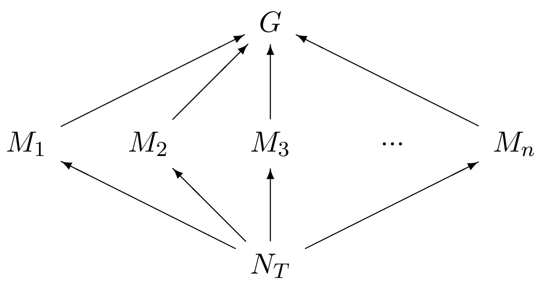

Figure 4.1: Amalgam formed from n maximal 2-local subgroups

$\mathcal{G}_2$ $T \in \mathrm{Syl}_2 P_i$;

$\mathcal{G}_3$ $T_G = 1$;

$\mathcal{G}_4$ $P_i/O_2(P_i) \cong S_3$; and

$\mathcal{G}_5$ $[\Omega(Z(T)), P_i] \neq 1$.

($O_2(P_i)$ is the p-radical as defined in Definition 2.14 and $\Omega(Z(T))$ is defined in Definition 2.13)

The aim will be to use the amalgam method to prove that assuming $\mathcal{G}$ implies:

Conclusion ($\mathcal{H}$)**.** G, P_1 and P_2 are such that either:

$\mathcal{H}_1$ $P_1 \cong P_2 \cong S_4$; or

$\mathcal{H}_2$ $P_1 \cong P_2 \cong C_2 \times S_4$.

So our aim is to prove the following theorem:

Theorem 4.1. *Let G be a group generated by two finite subgroups P_1 and P_2. If G, P_1 and P_2 satisfy $\mathcal{G}$ then $\mathcal{H}$ follows.*

4.3 Comparisons to the General Method

We can see how this example compares with the general case as given in [4, Chapter 28].

4.3.1 Amalgam Hypothesis

In the general case the Amalgam Hypothesis is assumed:

Assumption ($\mathcal{G}^*$)**.** Let X be a group with non-identity subgroups P, B, X_1 and X_2 and such that

$\mathcal{G}^*{}_1$ $X = \langle X_1, X_2 \rangle$;

$\mathcal{G}^*{}_2$ $B = X_1 \cap X_2$;

$\mathcal{G}^*{}_3$ No nonidentity subgroup of B is normal in both X_1 and X_2;

$\mathcal{G}^*{}_4$ P is a Sylow p-subgroup of both X_1 and X_2 for some prime p; and

$\mathcal{G}^*{}_5$ $F^*(X_i) = O_p(X_i)$ for $i = 1, 2$.

Where $F^*(X_i)$ is the generalized Fitting subgroup of X_i as defined below.

Definition 4.2. Let G be a finite group. We define the **Fitting subgroup**, $F(G)$ to be the unique largest normal nilpotent subgroup of G.

We define the **generalized Fitting subgroup**, $F^*(G)$ to be

$$\langle F(G), X \mid X \text{ a component of } G \rangle$$

where the components of G are the quasisimple subnormal subgroups of G.

Although $\mathcal{G}^*$ is a weaker set of assumptions than $\mathcal{G}$ a lot can still be inferred and generally the structure of X would be pinned down using extra assumptions, often involving $X_i/O_p(X_i)$ (such as $\mathcal{G}_4$).

In the general case, it is the structure of X_1 and X_2 that we are concerned with so X is defined to be the universal completion $\mathrm{gp}\langle \mathcal{D} \rangle$ (see Definition 5.9) to simplify matters. We therefore have an extra assumption

$\mathcal{G}^*{}_6$ $\mathcal{D} = \{B, X_1, X_2\}$ is a defining amalgam (see Definition 5.10).

We have analogues of results such as Lemma 3.3 and Lemma 3.4 as well as properties such as Γ being a tree (a graph without any circuits).

Often p is taken to be 2 which simplifies matters (for example, any group consisting only of elements of order 1 and 2 is Abelian).

4.4 Preliminaries

We now assume $\mathcal{G}$ at all times, and let Γ denote the coset graph of G with respect to G, P_1, P_2 and T.

Note that, by Lemma 3.2, Γ has only single edges between vertices and, by Theorem 3.6, Γ is connected. By Lemma 3.3(c) we have the kernel of the action of G on Γ being T_G and, by $\mathcal{G}_3$, we see that $T_G = 1$ and so G acts faithfully on Γ.

If we have an edge, $\{\alpha, \beta\}$, then it is conjugate to the edge $\{P_1, P_2\}$ and so the statements $\mathcal{G}_1, \ldots, \mathcal{G}_5$ hold for G_α and G_β instead of P_1 and P_2.

Lemma 4.3 ([7, 10.3.4]). *Let* $\{\alpha, \beta\} \in E(\Gamma)$.

(a) $G_\alpha \cap G_\beta$ *has index 3 in* G_β *and is a Sylow 2-subgroup of* G_β*. In particular* $G_\alpha = \langle G_\alpha \cap G_\beta, t \rangle$ *for all* $t \in G_\alpha \setminus G_\beta$.

(b) $|\Delta(\alpha)| = 3$ *and*

$$O_2(G_\alpha) = \bigcap_{\beta \in \Delta(\alpha)} (G_\alpha \cap G_\beta), \qquad (= G_\alpha^{[1]})$$

where $G_\alpha^{[1]}$ *is the group defined in Definition 2.35.*

(c) G_α *acts 2-transitively on* $\Delta(\alpha)$*.*

Proof. (a) follows from $\mathcal{G}_5$ and $\mathcal{G}_4$; (b) from Lemma 3.4; and (c) from Lemma 3.3(a).

□

We now denote two groups associated with each vertex $\alpha \in \Gamma$,

$Q_\alpha := O_2(G_\alpha)$; and
$Z_\alpha := \langle \Omega(Z(T)) \mid T \in \mathrm{Syl}_2(G_\alpha) \rangle$.;

where $\Omega(Z(T))$ is as defined in Definition 2.13.

In the general method Q_α and Z_α are defined slightly differently (however under $\mathcal{G}$ these definitions coincide). We have Q_α defined to be the kernel of the action of X_α on $\Delta(\alpha)$ and for P_α a Sylow p-subgroup of X_α we have

$$Z_\alpha := \langle \Omega(Z(P_\alpha))^{X_\alpha} \rangle.$$

Before we prove some elementary facts about Q_α and Z_α we need some precursory results.

Theorem 4.4 (Schur–Zassenhaus [7, 6.2.1])**.** *Let G be a group and K a normal subgroup of G such that $(|K|, |G/K|) = 1$. Then K has a complement in G. If in addition K or G/K is soluble, then all such such complements are conjugate in G.*

Proof. The proof is given in Appendix A under Theorem A.4.

□

Lemma 4.5 ([7, 8.4.2])**.** *Let the action of a group A on a group G be coprime. Then*

$$G = C_G(A) \times [G, A]$$

Proof. The proof is given in Appendix B under Lemma B.3. □

We can now prove some facts about Q_α and Z_α.

Lemma 4.6 ([7, 10.3.5])**.** *Let α be in $V(\Gamma)$, $V \trianglelefteq G_\alpha$ and T in $Syl_2(G_\alpha)$. Suppose that*

- $\Omega(Z(T)) \leq V \leq \Omega(Z(Q_\alpha))$*; and*

- $|V : \Omega(Z(T))| = 2$.

Then

$$V = C_V(G_\alpha) \times W, \quad \textit{where} \quad W := [V, G_\alpha].$$

Moreover $W \cong C_2 \times C_2$ *and* $C_{G_\alpha}(W) = Q_\alpha$, *i.e.* $G_\alpha / C_{G_\alpha}(W) \cong S_3$.

Proof. Let D be in $\mathrm{Syl}_3(G_\alpha)$. By Lemma 4.5 we have the decomposition

$$V = C_V(D) \times W, \quad \text{where} \quad W := [V, D].$$

Then $\mathcal{G}_5$ and the fact that $G_\alpha = DT$ imply that $W \neq 1$ and thus $|W| \geq 4$.
Let d be in $D^\#$. By our hypothesis,

$$|V/\Omega(Z(T))| = 2 = |V/\Omega(Z(T^d))|.$$

Now $G_\alpha = \langle T, T^d \rangle$ shows that $|V/C_V(G_\alpha)| \leq 4$. It follows that

$$C_V(G_\alpha) = C_V(D)$$

and $|W| = 4$. The other statements follow from $\mathcal{G}_4$.

□

Lemma 4.7 ([7, 10.3.6])**.** *Let* α, β *be incident in* Γ. *Then*

(a) $Z_\alpha \leq \Omega(Z(Q_\alpha))$;

(b) $Q_\alpha Q_\beta = G_\alpha \cap G_\beta \in \mathit{Syl}_2(G_\alpha)$;

(c) $C_{G_\alpha}(Z_\alpha) = Q_\alpha$; *in particular, the Sylow 2-subgroups of* G_α *are non-Abelian; and*

(d) $Z_\alpha Z_\beta$ *is normal in* G_α *if and only if there exists* γ *in* $\Delta(\alpha) \setminus \{\beta\}$ *such that* $Z_\alpha Z_\beta = Z_\alpha Z_\gamma$.

Proof. (a) Let T be in $\mathrm{Syl}_2(G_\alpha)$. Then Q_α is a 2-subgroup and so is contained in T, and $\mathcal{G}_1$ implies that $\Omega(Z(T)) \leq Z(Q_\alpha)$, so because Z_α is generated by the $\Omega(Z(T))$ it is a subgroup of $Z(Q_\alpha)$ and the result immediately follows.

(b) By $\mathcal{G}_4$ and Lemma 4.3 Q_α and Q_β have index 2 in $G_\alpha \cap G_\beta$. It is therefore enough to show that Q_α and Q_β are unequal.

For a contradiction, assume $Q_\alpha = Q_\beta$. Then as G acts faithfully on Γ we can apply Theorem 3.7 and Lemma 4.3 to see that $Q_\alpha = 1$. This contradicts $\mathcal{G}_1$ and so Q_α and Q_β are unequal and the result follows.

(c) By $\mathcal{G}_5$ the normal subgroup Z_α is not central in G_α. So Z_α is also not central in T a Sylow 2-subgroup of G_α since this would contradict $\mathcal{G}_5$ as $G_\alpha = \langle T \mid T \in \mathrm{Syl}_2(G_\alpha)\rangle$.

By part (a) we have $Q_\alpha \leq C_{G_\alpha}(Z_\alpha)$. If Q_α and $C_{G_\alpha}(Z_\alpha)$ were unequal then $C_{G_\alpha}(Z_\alpha)$ would contain a subgroup D of order 3, and $G_\alpha = DT$, with T a Sylow 2-subgroup of G_α, by $\mathcal{G}_4$. But then we would have $\Omega(Z(T))$ being central in G_α, which contradicts $\mathcal{G}_5$.

(d) If $Z_\alpha Z_\beta \trianglelefteq G_\alpha$, then $Z_\alpha Z_\beta = Z_\alpha Z_\gamma$ for all γ in $\Delta(\alpha)$ since G_α is transitive on $\Delta(\alpha)$ so the forward implication is proven.

Assume now that $Z_\alpha Z_\beta = Z_\alpha Z_\gamma$ for some γ in $\Delta(\alpha)$ not equal to β. Then $Z_\alpha Z_\beta$ is normalized by $G_\alpha \cap G_\beta$ and thus, by $\mathcal{G}_4$, it is also normalized by G_α and so the equivalence holds.

□

4.5 The Main Concept

We now start to create the links between the coset graph Γ and our preferred conclusion $\mathcal{H}$.

Lemma 4.8. *Let α and β be adjacent in Γ and let δ be in $\{\alpha, \beta\}$. Assume we have*

$$G_\delta \cong S_4 \quad \textit{and} \quad Q_\delta \cong C_2 \times C_2$$

or

$$G_\delta \cong S_4 \times C_2 \quad \textit{and} \quad Q_\delta \cong C_2 \times C_2 \times C_2.$$

Then $Z_\delta = Q_\delta$.

Proof. Consider the first case: we know from Sylow's Theorems and the fact $|S_4| = 3 \times 2^3$ that S_4 has either 1 or 3 Sylow 2-subgroups each of size 8. With a little calculation we can see that there are in fact 3 (all conjugate) and they are:

$A = \{(1234), (1432), (13), (24), (12)(34), (13)(24), (14)(23), e\}$;

$B = \{(1243), (1342), (14), (23), (12)(34), (13)(24), (14)(23), e\}$; and

$C = \{(1324), (1423), (12), (34), (12)(34), (13)(24), (14)(23), e\}$.

As they all intersect in V_4 and $V_4 \leq Z(V_4)$ we initially consider whether any elements of V_4 are in the centralizers of A, B or C. We see that

$$\begin{aligned} (13)(24) &\in Z(A); \\ (14)(23) &\in Z(B); \text{ and} \\ (12)(34) &\in Z(C). \end{aligned}$$

As $Z(A)$, $Z(B)$ and $Z(C)$ are all 2-groups we need only consider the remaining involutions and elements of order 4 in S_4 to calculate the elements of the centralizers. It is clear by considering any element (ij) and any element $(ijkl)$ that elements of either form are not members of the centralizers.

We therefore have:

$$\begin{aligned} Z(A) &= \{e, (13)(24)\} = \Omega(Z(A)); \\ Z(B) &= \{e, (14)(23)\} = \Omega(Z(B)); \text{ and} \\ Z(C) &= \{e, (12)(34)\} = \Omega(Z(C)). \end{aligned}$$

And so we get

$$\begin{aligned} Z_\delta &= \langle \Omega(Z(T)) \mid T \in \mathrm{Syl}_2(G_\delta) \rangle \\ &= \langle \Omega(Z(A)), \Omega(Z(B)), \Omega(Z(C)) \rangle \\ &= \langle e, (12)(34), (13)(24), (14)(23) \rangle \\ &= V_4 \;=\; Q_\delta. \end{aligned}$$

The second result easily follows from the first, whence $Z_\delta = Q_\delta = V_4 \times C_2$. □

Theorem 4.9 ([7, 10.3.7])**.** *Let α and β be adjacent in Γ. The following statements are then equivalent:*

(i) $\mathcal{H}$ holds.

(ii) $Z_\alpha \nleq Q_\beta$

Proof. First, assume that $\mathcal{H}$ holds. Then for δ in $\{\alpha, \beta\}$ we have:

$$G_\delta \cong S_4 \quad \text{and} \quad Q_\delta \cong C_2 \times C_2;$$

or

$$G_\delta \cong S_4 \times C_2 \quad \text{and} \quad Q_\delta \cong C_2 \times C_2 \times C_2.$$

We can therefore use Lemma 4.8 to see that $Z_\delta = Q_\delta$, and by Lemma 4.7(b) we have $Z_\alpha \nleq Q_\beta$, so statement (i) implies statement (ii).

Assume now that $Z_\alpha \nleq Q_\beta$. Let δ be in $\{\alpha, \beta\}$ and set

$$T := Q_\alpha Q_\beta \quad \text{and} \quad E := Q_\alpha \cap Q_\beta.$$

Once again, using Lemma 4.7(b), we see that T is a Sylow 2-subgroup of G_δ and $|T/Q_\delta| = 2$. We therefore get

$$|Q_\alpha : E| = 2 = |Q_\beta : E|, \tag{4.1}$$

and

$$T = Q_\beta Z_\alpha \quad \text{and} \quad Q_\alpha = EZ_\alpha. \tag{4.2}$$

Now from Lemma 4.7(c) we see that $[Z_\alpha, Z_\beta] \neq 1$ and so we also have

$$Z_\beta \nleq Q_\alpha.$$

Using symmetry and (4.1) we see that

$$T = Q_\alpha Z_\beta \quad \text{and} \quad Q_\beta = EZ_\beta. \tag{4.3}$$

Now recall Definition 2.3 of $\Phi(G)$. We know that Z_δ is an elementary Abelian subgroup of $Z(Q_\delta)$ so using (4.2) and (4.3) with Corollary 2.6 we see that

$$\Phi(Q_\alpha) = \Phi(E) = \Phi(Q_\beta).$$

Hence we have $\Phi(E)$ characteristic in Q_δ. We therefore have $\Phi(E)$ normal in both G_α and G_β. We now apply Theorem 3.7 and see that $\Phi(E)$ must be trivial. We now have

$$Q_\alpha \text{ and } Q_\beta \text{ are elementary Abelian.} \tag{4.4}$$

Also, the fact $T = Q_\alpha Q_\beta$ means that

$$E = Z(T). \tag{4.5}$$

Now denote $W_\delta := [Q_\delta, G_\delta]$. Then we can use (4.1) to apply Lemma 4.6 with $V = Q_\delta$, so

$$Q_\delta = Z(G_\delta) \times W_\delta \quad \text{and} \quad W_\delta \cong C_2 \times C_2. \tag{4.6}$$

By (4.2) and (4.3) there exists an involution t_δ in T/Q_δ that acts non-trivially on $O^2(G_\delta)/W_\delta$. Hence

$$X_\delta := O^2(G_\delta)\langle t_\delta \rangle \cong S_4.$$

We now consider the two cases of $Z(G_\alpha) = 1$ and $Z(G_\alpha) \neq 1$ separately.

First we assume that $Z(G_\alpha) = 1$. Then $|T| = 8$, and so $Z(G_\beta) = 1$ follows from (4.5) and (4.6). We therefore have

$$G_\alpha = X_\alpha \qquad \text{and} \qquad G_\beta = X_\beta$$

as in $\mathcal{H}_1$.

Now consider the second case, where $Z(G_\alpha) \neq 1$. We therefore have, from (4.5) and (4.6), that $Z(G_\beta) \neq 1$. But from Theorem 3.7 we know

$$Z(G_\alpha) \cap Z(G_\beta) = 1.$$

As $Z(G_\alpha)$ and $Z(G_\beta)$ are in $Z(T) = T$ we can see from (4.6) that

$$Z(G_\alpha) \cong C_2 \cong Z(G_\beta).$$

It is clear that $\mathcal{H}_2$ follows immediately.

□

Therefore Theorem 4.1 is equivalent to saying that $\mathcal{G}$ implies for all adjacent vertices $Z_\alpha \nleq Q_\beta$. We will now therefore assume there exists a pair of incident vertices with $Z_\alpha \leq Q_\beta$ and show this leads to a contradiction. We do this using a new concept, that of a critical pair.

Definition 4.10. Define the **critical pair constant**, b, as follows:

$$b := \min\{d(\alpha, \beta) \mid \alpha, \beta \in \Gamma, Z_\alpha \nleq Q_\beta\}.$$

Now define a pair (α, α') of vertices to be a **critical pair** if

$$Z_\alpha \nleq Q_{\alpha'} \quad \text{and} \quad d(\alpha, \alpha') = b.$$

Note. If α is a vertex of Γ then because Z_α acts faithfully on Γ there must exist a vertex β with $Z_\alpha \nleq G_\beta$. Hence $Z_\alpha \nleq Q_\beta$ and Γ is connected by Theorem 3.6 so $d(\alpha, \beta) < \infty$ and b is an integer.

In the general amalgam method critical pairs are defined identically and play just as crucial a role.

Now if α and β are vertices of Γ with $d(\alpha, \beta) < b$ then by the minimality of b we have

$$Z_\alpha \leq Q_\beta \quad \text{and} \quad Z_\beta \leq Q_\alpha.$$

Therefore, by Theorem 4.9, Theorem 4.1 is equivalent to saying $\mathcal{G}$ implies $b = 1$ for Γ.

We now consider some properties of critical pairs. We use the following notation; let (α, α') be a critical pair and γ a path from α to α' of length b. Enumerate the vertices of γ by

$$\gamma = (\alpha, \alpha+1, \alpha+2, \dots, \alpha') \quad \text{or} \quad \gamma = (\alpha, \dots, \alpha'-2, \alpha'-1, \alpha'),$$

with the obvious relation

$$\alpha' - i = \alpha + (b - i) \quad \text{for} \quad 1 \leq i \leq b-1.$$

We also denote

$$R := [Z_\alpha, Z_{\alpha'}].$$

Lemma 4.11 ([7, 10.3.8])**.** *Let (α, α') be a critical pair of Γ. Then*

(a) (α', α) *is also a critical pair;*

(b) $G_\alpha \cap G_{\alpha+1} = Z_{\alpha'} Q_\alpha$ *and* $G_{\alpha'-1} \cap G_{\alpha'} = Z_\alpha Q_{\alpha'}$*;*

(c) $R \leq Z(G_\alpha \cap G_{\alpha+1}) \cap Z(G_{\alpha'-1} \cap G_{\alpha'})$ *and* $R = [Z_\alpha, G_{\alpha+1} \cap G_\alpha] = [Z_{\alpha'}, G_{\alpha'-1} \cap G_{\alpha'}]$*;*

(d) $|R| = 2$*;*

(e) $Z_\alpha = [Z_\alpha, G_\alpha] \times \Omega(Z(G_\alpha))$ *and* $[Z_\alpha, G_\alpha] \cong C_2 \times C_2$*; and*

(f) $|Z_\alpha : \Omega(Z(Y))| = 2$ *for all Sylow 2-subgroups* Y *of* G_α.

Proof. We first use the minimality of B to see

$$Z_\alpha \leq Q_{\alpha'-1} \leq G_{\alpha'-1} \cap G_{\alpha'} \quad \text{and} \quad Z_{\alpha'} \leq Q_{\alpha+1} \leq G_\alpha \cap G_{\alpha+1}$$

The fact that $Z_\alpha \nleq Q_{\alpha'}$ provides us with

$$G_{\alpha'-1} \cap G_{\alpha'} = Z_\alpha Q_{\alpha'}$$

since by Lemma 4.3 and $\mathcal{G}_4$, $Q_{\alpha'}$ has index 2 in $G_{\alpha'-1} \cap G_{\alpha'}$.
Now as Z_α and $Z_{\alpha'}$ are normal in G_α and $G_{\alpha'}$, respectively, we see that

$$R \leq Z_\alpha \cap Z_{\alpha'}. \tag{4.7}$$

We now apply Lemma 4.7(c) and see that $R \neq 1$, so we we also have

$$Z_{\alpha'} \nleq Q_\alpha \quad \text{and} \quad G_\alpha \cap G_{\alpha+1} = Z_{\alpha'} Q_\alpha$$

We therefore have part (a) (as $d(\alpha, \alpha') = d(\alpha', \alpha)$) and part (b). We use (4.7) and Lemma 4.7(a) to prove part (c). We now use Lemma 4.7(a) and (c) to see that

$$|Z_\alpha / C_{Z_\alpha}(Z_{\alpha'})| = |Z_{\alpha'} / C_{Z_{\alpha'}}(Z_\alpha)| = 2 \quad \text{and} \quad C_{Z_\alpha}(Z_{\alpha'}) = \Omega(Z(G_\alpha \cap G_{\alpha+1})). \tag{4.8}$$

We use (4.8) to see part (d) and part (f), and finally Lemma 4.6 implies part (e).

□

In the general method it turn out that (a) is only true if $[Z_\alpha, Z_\beta] \neq 1$ and $Q_\alpha = O_p(X_\alpha)$.

We now prove some important results about vertices in $\Delta(\alpha)$.

Lemma 4.12 ([7, 10.3.9])**.** *Let* $\alpha - 1$ *be in* $\Delta(\alpha) \backslash \{\alpha + 1\}$ *such as in Figure 4.2. Suppose that* $(\alpha - 1, \alpha' - 1)$ *is not a critical pair. Then we have the following results:*

(a) $Z_\alpha Z_{\alpha+1} = Z_\alpha Z_{\alpha-1} \trianglelefteq G_\alpha$*;*

(b) $Q_\alpha \cap Q_\beta \trianglelefteq G_\alpha$ *for all* β *in* $\Delta(\alpha)$*; and*

(c) α *and* α' *are conjugate, and* b *is even.*

Proof. Since we have $(\alpha - 1, \alpha' - 1)$ is not critical we have the following

$$Z_{\alpha-1} \leq Q_{\alpha'-1} \quad (\leq G_{\alpha'-1} \cap G_{\alpha'}).$$

In particular $b > 1$ and

$$Z_{\alpha-1} \leq Z_\alpha Q_{\alpha'} = Z_\alpha C_{G_{\alpha'}}(Z_{\alpha'}).$$

Where the first inequality follows from Lemma 4.11 and the equality from Lemma 4.7. We can therefore infer

$$[Z_{\alpha-1}, Z_{\alpha'}] \leq R \leq Z_\alpha,$$

where the second inequality follows from Lemma 4.7 again.

We therefore have $Z_{\alpha-1}Z_\alpha$ being normalized by $Z_{\alpha'}$ and $G_{\alpha-1} \cap G_\alpha$ and so, by Lemma 4.3, also by $\langle G_\alpha \cap G_{\alpha-1}, Z_{\alpha'} \rangle = G_\alpha$. We now use Lemma 3.4 to see (a).

To obtain part (b) we use part (a) along with Lemma 4.7(c) and Lemma 3.4 again.

Now we have two cases: either α is in $(\alpha')^G$ or α is in $(\alpha'-1)^G$, so

$$\alpha \in (\alpha')^G \iff b \text{ is even.}$$

To finish proving (c) we may assume that α and $\alpha'-1$ are conjugate, so we also have G_α and $G_{\alpha'-1}$ conjugate. We then apply (b) to see

$$Z_\alpha \leq Q_{\alpha'-2} \cap Q_{\alpha'-1} = Q_{\alpha'-1} \cap Q_{\alpha'}.$$

However, from this we see that $Z_\alpha \leq Q_{\alpha'}$ contradicting our assumption that (α, α') is a critical pair.

□

Theorem 4.13 ([7, 10.3.10])**.** *Let (α, α') be a critical pair. Assume there exists a vertex $\alpha - 1$ in $\Delta(\alpha) \backslash \{\alpha + 1\}$ such that $(\alpha - 1, \alpha' - 1)$ is a critical pair. Then $b = 1$.*

Proof. Label vertices as shown in Figure 4.2, so that $\alpha-2$ is in $\Delta(\alpha-1)\backslash\{\alpha\}$.

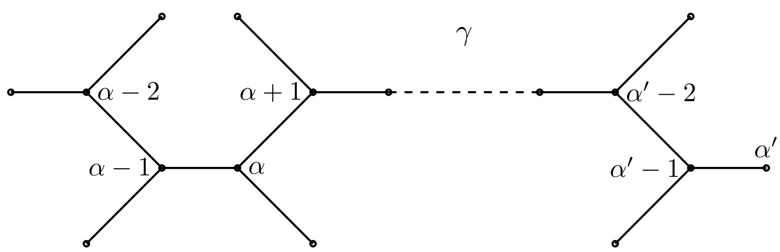

Figure 4.2: Sample path γ [7, pp. 291 Fig. 1]

Now denote

$$R_1 := [Z_{\alpha-1}, Z_{\alpha'-1}]$$

We assume that $b > 1$ and derive a contradiction.

We know that because $d(\alpha, \alpha+1) < b$ and $d(\alpha', \alpha'-1) < b$ we have $Z_\alpha \leq Q_{\alpha+1}$ and $Z_{\alpha'} \leq Q_{\alpha'-1}$. Using the assumption that $(\alpha-1, \alpha'-1)$ is critical we apply Lemma 4.11 to this pair to see that $|R_1| = 2$ and

$$R_1 = [Z_{\alpha-1}, G_{\alpha-1} \cap G_\alpha] \leq Z(G_{\alpha-1} \cap G_\alpha) \cap Z(G_{\alpha'-2} \cap G_{\alpha'-1}).$$

In particular we have that $R_1 \leq Z(Q_{\alpha'-1})$ and so $[R_1, Z_{\alpha'}] = 1$. We now apply Lemma 4.11(b) to see that $Z_{\alpha'}$ and $G_{\alpha-1} \cap G_\alpha$ generate G_α, so we have

$$R_1 \leq Z(G_\alpha). \tag{4.9}$$

Now we show:

$$(\alpha-2, \alpha'-2) \text{ is a critical pair.} \tag{4.10}$$

Assume, for a contradiction that $(\alpha-2, \alpha'-2)$ is not a critical pair. Then we can apply 4.12(a) to the pair $(\alpha-1, \alpha'-1)$ and vertex $\alpha-2$ to see that for all δ in $\Delta(\alpha-1)$ we have $Z_{\alpha-1}Z_\alpha = Z_{\alpha-1}Z_\delta$.

Now consider the vertices around α and rotate them so that $\alpha-1$ goes to $\alpha+1$. We can then apply 3.4 to see that

$$Z_{\alpha+1}Z_\alpha = Z_{\alpha+1}Z_{\alpha+2}.$$

We now appeal to the minimality of b to see that $Z_{\alpha+1}Z_{\alpha+2} \leq Q_{\alpha'}$ and it follows that $Z_\alpha \leq Q_{\alpha'}$. This implies that (α, α') is not a critical pair, which is the desired contradiction to prove (4.10) holds.

Now let $R_2 := [Z_{\alpha-2}, Z_{\alpha'-2}]$. We now use the recently proven (4.10) to see that the pair (α, α') and the vertex $\alpha-2$ also satisfies the hypothesis. We therefore get that $|R_2| = 2$ and

$$R_2 = [Z_{\alpha-2}, G_{\alpha-2} \cap G_{\alpha-1}] \leq Z(G_{\alpha-1}). \tag{4.11}$$

Now by 3.4 there exists $y \in G_{\alpha-1}$ and $x \in G_\alpha$ such that

$$(\alpha-2)^y = \alpha \qquad \text{and} \qquad (\alpha+1)^x = \alpha-1.$$

Hence

$$[Z_\alpha, G_\alpha \cap G_{\alpha-1}] = [Z_{\alpha-2}, G_{\alpha-2} \cap G_{\alpha-1}]^y = {R_2}^y \leq Z(G_{\alpha-1}),$$

and using 4.11(c) we see also

$$R^x = [Z_\alpha, G_\alpha \cap G_{\alpha+1}]^x = [Z_\alpha, G_\alpha \cap G_{\alpha-1}] = {R_2}^y \leq Z(G_{\alpha-1}).$$

It therefore follows that

$$R \leq Z(G_{\alpha+1}). \tag{4.12}$$

We also have, applying (4.9) and 3.7,

$$R \cap R_1 = 1. \tag{4.13}$$

Now we show that it must be the case that if $b > 1$ we have

$$b = 2. \tag{4.14}$$

Assume the contrary, that $b > 2$. Then $Z_{\alpha'} \leq Q_{\alpha'-2}$, and by (4.11) and Lemma 4.7(a) we have that R_2 centralizes $Z_{\alpha'}$ and $G_{\alpha-1}$. Since

$$G_\alpha = \langle Z_{\alpha'}, G_\alpha \cap G_{\alpha-1} \rangle,$$

we must have the fact that R_2 centralizes $G_{\alpha-1}$ and G_α. We can now apply Theorem 3.7 to see that $R_2 = 1$, which contradicts the fact that $|R_2| = 2$, and so (4.14) holds.

We now treat the remaining case of $b = 2$. We first denote:

$$V_\alpha := \langle Z_\beta \mid \beta \in \Delta(\alpha) \rangle, \qquad (\trianglelefteq G_\alpha)$$

and

$$V_{\alpha+1} := \langle Z_\beta \mid \beta \in \Delta(\alpha+1) \rangle. \qquad (\trianglelefteq G_{\alpha+1})$$

Now because $b > 1$ we have $Z_\beta \leq Q_\alpha$ for all β in $\Delta(\alpha)$ and also $Z_\gamma \trianglelefteq Q_{\alpha+1}$ for all γ in $\Delta(\alpha+1)$ so $V_\alpha \leq Q_\alpha$ and $V_{\alpha+1} \leq Q_{\alpha+1}$. We also have

$$Z_\alpha = \langle \Omega(Z(G_\alpha \cap G_{\alpha+1}))^{G_\alpha} \rangle \leq V_\alpha,$$

because V_α is normal in G_α, and similarly, because $V_{\alpha+1}$ is normal in $G_{\alpha+1}$, we have $Z_{\alpha+1} \leq V_{\alpha+1}$. We can therefore see that

$$Z_\alpha Z_{\alpha+1} \leq V_\alpha \cap V_{\alpha+1}. \tag{4.15}$$

Now because we know $R_1 \leq Z(G_\alpha$, the 2-transitive action of G_α on $\Delta(\alpha)$ (Lemma 4.3(c)) gives us

$$V'_\alpha = R_1 \leq Z(G_\alpha).$$

We now derive a contradiction showing that V_α is Abelian: Since V_α is generated by involutions we have V_α/R_1 elementary Abelian so

$$R_1 = \Phi(V_\alpha),$$

where Φ is the Frattini subgroup as defined in Definition 2.3. We now use (4.12) to see

$$R = \Phi(V_{\alpha+1}).$$

Now denote

$$\overline{V}_\alpha := V_\alpha / Z_\alpha.$$

We can use Lemma 4.11(f) to get $|Z_\beta / Z_\alpha \cap Z_\beta| = 2$ for all β in $\Delta(\alpha)$, so we must have $|\overline{Z}_\beta| = 2$. In addition, $\overline{V}_\alpha$ is generated by the three subgroups $\overline{Z}_\beta$ with β in $\Delta(\alpha)$, so we have

$$|\overline{V}_\alpha| \leq 8. \tag{4.16}$$

Now set

$$W := V_\alpha \cap V_{\alpha+1}. \qquad (\trianglelefteq G_\alpha \cap G_{\alpha+1})$$

We use (4.15) to see that $Z_\alpha Z_{\alpha+1} \leq W$, and the definition of V_α gives us

$$V_\alpha = \langle W^{G_\alpha} \rangle. \tag{4.17}$$

We now apply Lemma 2.7 to see

$$\Phi(W) \leq \Phi(V_\alpha) \cap \Phi(V_{\alpha+1}) = R_1 \cap R = 1,$$

where the final equality follows from (4.13). We therefore have the fact W is elementary Abelian, and $V'_\alpha \neq 1$ shows that $|V_\alpha/W| \geq 2$.

We now look at the action of G_α on $\overline{V}_\alpha$. The kernel of this action contains Q_α since $[G_{\alpha-1} \cap G_\alpha, Z_{\alpha-1}] = R_1 \leq Z_\alpha$. Now denote

$$\overline{V}_0 := [\overline{V}_\alpha, O^2(G_\alpha)].$$

First consider the case that $\overline{V}_0 = 1$. Then W is normal in G_α and $V'_\alpha = 1$. But this contradicts the result that $V'_\alpha = R_1$ as we have also proven $|R_1| = 2$.

Now consider the other case, that $\overline{V}_0 \neq 1$. We now use (4.16) we see that

$$|\overline{V}_0| = 4. \tag{4.18}$$

Now assume $|V_\alpha/W| = 2$. Let x be in G_α such that $W^x \neq W$. Then $V_\alpha = WW^x$ and so $W \cap W^x = Z(V_\alpha)$ and $|V_\alpha/W \cap W^x| = 4$. Let D be in $\mathrm{Syl}_3(G_\alpha)$. The nontrivial action of D on $\overline{V}_\alpha$ implies a nontrivial actiuon of D on $V_\alpha/W \cap W^x$. Therefore, all maximal subgroups of V_α that contain $W \cap W^x$ are D-conjugates of W. However, this would mean every element of $V_\alpha^\#$ is an involution and so V_α is elementary Abelian, but this contradicts the fact $V'\alpha = R_1$.

We therefore have that

$$|V_\alpha/W| \geq 4.$$

We can now apply (4.15) and use (4.16) to see

$$|\overline{V}_\alpha| = 8, \quad W = Z_\alpha Z_{\alpha+1} \quad \text{and} \quad |\overline{W}| = 2. \tag{4.19}$$

Now because $Z_{\alpha'} \leq G_\alpha$ and $Z_{\alpha'} \not\leq Q_\alpha$ we see that $[\overline{V}_o, Z_{\alpha'}] \neq 1$. However, we also see that because $b = 2$ we have

$$[V_\alpha, Z_{\alpha'}] \leq [V_\alpha, V_{\alpha+1}] \leq W,$$

so $\overline{W} = [\overline{V}_0, Z_{\alpha'}]$. But then

$$\langle \overline{W}^{G_\alpha} \rangle = \overline{V}_0,$$

which contradicts (4.17), (4.18) and (4.19), so we have $b = 1$ as desired.

□

4.6 Main Result

We are now in a position to prove Theorem 4.1:

Theorem 4.14 ([7, 10.3.11]). *Suppose $\mathcal{G}$ holds. Then P_1 and P_2 satisfy $\mathcal{H}$; that is, either*

$$P_1 \cong P_2 \cong S_4 \qquad \text{or} \qquad P_1 \cong P_2 \cong C_2 \times S_4.$$

Proof. Let G satisfy $\mathcal{G}$ but not $\mathcal{H}$. Among all the quadruples (G, P_1, P_2, T) that satisfy $\mathcal{G}$ but not $\mathcal{H}$ choose the quadruple such that $|T|$ is minimal. Then by Theorem 4.9 we obtain that $b > 1$ and by Theorem 4.13 $(\alpha - 1, \alpha' - 1)$ is not critical for all $\alpha - 1$ in $\Delta(\alpha)/\{\alpha + 1\}$. We now apply Lemma 4.12 to see

$$b \equiv 0 (\text{mod } 2) \quad \text{and} \quad X := Q_\alpha \cap Q_{\alpha+1} \trianglelefteq G_\alpha. \tag{4.20}$$

We can also apply Theorem 3.7(b) and Lemma 4.3(a) to get

$$|Q_\alpha : X| = |Q_{\alpha+1} : X| = 2.$$

Now let

$$D \in \mathrm{Syl}_3(G_\alpha) \qquad \text{and} \qquad \overline{G}_\alpha := G_\alpha / X.$$

Then $\overline{G}_\alpha$ is a group of order 12 and $\overline{Q}_\alpha$ is a normal subgroup of order 2. It therefore follows that $\overline{D}$ is also normal in $\overline{G}_\alpha$. Now let $X \leq L \leq G_\alpha$ such that $\overline{L} = \overline{DQ}_{\alpha+1}$. We now obtain the following results

$$L \text{ is a normal subgroup of index 2 in } G_\alpha; \tag{4.21}$$
$$\overline{L} \cong S_3; \tag{4.22}$$
$$\mathrm{Syl}_2(L) = \{Q_\beta \mid \beta \in \Delta(\alpha)\}; \tag{4.23}$$
$$O_2(L) = X = Q_\alpha \cap Q_\beta \text{ for all } \beta \in \Delta(\alpha); \tag{4.24}$$
$$Q_{\alpha+1} = Z_{\alpha'} O_2(L); \text{ and} \tag{4.25}$$
$$C_L(O_2(L)) \leq O_2(L). \tag{4.26}$$

Where (4.25) follows from Lemma 4.11(b) and to see (4.26) note that Z_α ($\trianglelefteq G_\alpha$) is contained in $Q_{\alpha+1}$ and thus also in $O_2(L)$. We therefore see

$$C_L(O_2(L)) \leq C_L(Z_\alpha) \leq Q_\alpha \cap L \leq O_2(L),$$

where the central inequality is from Lemma 4.7.

Now $\mathcal{G}_4$ tells us there exists an element t in $G_{\alpha+1}/Q_{\alpha+1}$ such that

$$\alpha^t = \alpha + 2 \quad \text{and} \quad t^2 \in Q_{\alpha+1}.$$

Therefore $Q_{\alpha+1} = (Q_{\alpha+1})^t$ is a Sylow 2-subgroup of L ($\leq G_\alpha$) and L^t ($\leq G_{\alpha+2}$). First we show:

$$O_2(L) \text{ is not elementary Abelian.} \tag{4.27}$$

Assume that $O_2(L)$ is a counterexample to this. Then by (4.22), (4.24) we can denote

$$A_1 := O_2(L) \quad \text{and} \quad A_2 := O_2(L^t).$$

A_1 and A_2 are two elementary Abelian subgroups of index 2 in $Q_{\alpha+1}$. Consider the two cases $A_1 = A_2$ and $A_1 \neq A_2$.

If A_1 and A_2 are equal then A_1 must be normal in $\langle G_\alpha, G_{\alpha+2}\rangle$ and so slao normal in

$$\langle G_\alpha, G_\alpha \cap G_{\alpha+1}, G_{\alpha+1} \cap G_{\alpha+2}\rangle = \langle G_\alpha, G_{\alpha+1}\rangle = G.$$

But this contradicts $\mathcal{G}_3$ and (4.26).

We therefore have A_1 not equal to A_2. We can use (4.22), (4.23) and (4.26) to see that $Q_{\alpha+1}$ is non-Abelian and so we have

$$A := A_1 \cap A_2 = Z(Q_{\alpha+1}) \quad \text{and} \quad |Q_{\alpha+1}/A| = 4.$$

Now if $O^2(G_{\alpha+1})$ acted trivially on $Q_{\alpha+1}/A$ then we would have

$$\langle G_\alpha, O^2(G_{\alpha+1})\rangle \leq N_G(A_1),$$

Which contradicts Theorem 3.7. We therefore must have $O^2(G_{\alpha+1})$ acting transitively on $(Q_{\alpha+1}/A)^\#$. But that means every element of $Q^\#_{\alpha+1}$ is an involution and $Q_{\alpha+1}$ is elementary Abelian, again a contradiction. We therefore have (4.27) holding.

Now we denote

$$G_0 := \langle L, L^t\rangle$$

and we denote the largest normal subgroup of G_0 in $Q_{\alpha+1}$ by Q. Because we have $G_0^t = G_0$ we also have $Q^t = Q$. Now we want to show that

$$[Q, D] \neq 1. \tag{4.28}$$

To prove this, assume that $[Q, D] = 1$ and set

$$\widetilde{G}_0 := G_0/Q.^\dagger$$

We now use the fact $Q_{\alpha+1}$ is a Sylow 2-subgroup of both L and L^t and (4.21)-(4.26) to see that the quadruple

$$(\widetilde{G}_0, \widetilde{L}, \widetilde{L}^t, \widetilde{Q}_{\alpha+1})$$

† In this proof we use the tilde notation, $\widetilde{G}_0$, rather than the bar notation $\overline{G}_0$ to follow convention.

satisfies our hypotheses $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$. Now as we are assuming $[Q, D] = 1$ we can use Lemma 4.7(c) and Corollary B.2 to see

$$\widetilde{W} := [\widetilde{Z}_\alpha, \widetilde{D}] \neq 1 \qquad (Q \leq W \leq O_2(L)),$$

and also

$$C_{\widetilde{L}}(O_2(\widetilde{L})) \leq O_2(\widetilde{L})$$

so we have $\mathcal{G}_1$ and $\mathcal{G}_2$ also holding and so all of $\mathcal{G}$ is satisfied.

We now appeal to the minimality of $|T|$ to establish a contradiction and so prove (4.28). As $|Q_{\alpha+1}| < |T|$ we get that either

$$\widetilde{L} \cong S_4 \qquad \text{or} \qquad \widetilde{L} \cong C_2 \times S_4.$$

So by Lemma 4.6 we have

$$\widetilde{W} = [O_2(\widetilde{L}), O^2(\widetilde{L})] \not\leq O_2(\widetilde{L}^t),$$

and $\widetilde{W} \leq \widetilde{Z}_\alpha$ implies $Z_\alpha \not\leq O_2(L^t)$. Now using (4.22) we have

$$O_2(L) = (O_2(L) \cap O_2(L^t))Z_\alpha$$

Since $Z_\alpha \leq \Omega(Z(O_2(L)))$ we get

$$\Phi(O_2(L)) = \Phi(O_2(L) \cap O_2(L^t)),$$

and if we conjugate with t we see that $\Phi(O_2(L)) = \Phi(O_2(L^t))$. But now – as in the proof of step (4.27) – $\Phi(O_2(L))$ is normal in $\langle G_\alpha, G_{\alpha+2}\rangle = G$, and $\mathcal{G}_3$ gives us that $\Phi(O_2(L)) = 1$. This contradicts (4.27) and so we have established (4.28).

We now show

$$\text{Let } \beta \in \Delta(\alpha) \text{ and } \gamma \in \Delta(\beta)\backslash\{\alpha\}. \quad \text{Then } \langle Z_\alpha, Z_\gamma\rangle \text{ is not normal in } L. \tag{4.29}$$

We fix the notation

$$\Delta(\beta) = \{\alpha, \gamma, \delta\},$$

and we set

$$V_\beta := \langle Z_\alpha, Z_\gamma, Z_\delta\rangle \qquad (\trianglelefteq G_\beta).$$

Now every x in $Q_\alpha \backslash Q_\beta$ interchanges γ and δ and normalizes L (from (4.21)). Now if $\langle Z_\alpha, Z_\gamma\rangle$ is normal in L, then we also have $\langle Z_\alpha, Z_\delta\rangle$ equal to $\langle Z_\alpha, Z_\gamma^x\rangle$, and so is normal in L. This implies that V_β is normal in L ($\not\leq G_\alpha \cap G_\beta$), which contradicts Theorem 3.7. This shows (4.29) and we now show

$$\text{Let } b \geq 4, \alpha - 1 \in \Delta(\alpha)\backslash\{\alpha+1\} \text{ and } \alpha - 2 \in \Delta(\alpha-1)\backslash\{\alpha\}. \quad \text{Then } (\alpha-2, \alpha'-2) \text{ is a critical pair.} \tag{4.30}$$

To prove this we assume that $(\alpha-2,\alpha'-2)$ is not critical. Then $Z_{\alpha-2} \leq Q_{\alpha'-3} \cap Q_{\alpha'-2}$. Since by (4.20) $\alpha'-2$ is conjugate to α we get

$$\begin{aligned} Z_{\alpha-2} &\leq Q_{\alpha'-3} \cap Q_{\alpha'-2} & \\ &= Q_{\alpha'-2} \cap Q_{\alpha'-1} & \text{from (4.20)} \\ &\leq G_{\alpha'-1} \cap G_{\alpha'} & \\ &= Z_\alpha Q_{\alpha'}, & \text{from 4.11(b)} \end{aligned}$$

and so we have

$$[Z_{\alpha-2}, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha.$$

Hence $Z_{\alpha-2}Z_\alpha$ $(\leq Q_\alpha \cap Q_{\alpha-1})$ is normalized by $Z_{\alpha'}$ and also by $Q_{\alpha-1}$ $(\leq G_{\alpha-2} \cap G_\alpha)$. Now (4.21) - (4.26) imply that $Z_{\alpha-2}Z_\alpha$ is normal in L which contradicts 5 and so (4.30) is proved.

Now for the following argument let $\alpha-1$ be in $\Delta(\alpha)\backslash\{\alpha+1\}$ and let x be in L $(\leq G_\alpha)$ such that

$$(\alpha+1)^x = \alpha-1.$$

Then

$$\alpha-2 := (\alpha+2)^x$$

is adjacent to $\alpha-1$ and is different from both α and $\alpha+2$.

Now we will show

$$b \leq 4. \qquad (4.31)$$

First we assume $b \geq 4$. Applying (4.30) we see $(\alpha-2,\alpha'-2)$ is a critical pair. We can therefore use Lemma 4.11 to see

$$R_2 := [Z_{\alpha-2}, Z_{\alpha'-2}] \leq Z(G_{\alpha-2} \cap G_{\alpha-1}) \cap Z_{\alpha'-2}.$$

In addition to this, $b \geq 4$ tells us $Z_{\alpha'} \leq Q_{\alpha'-2}$, so we also have $[R_2, Z_{\alpha'}] = 1$. Now we can use (4.21) - (4.26) to give us $[R_2, L] = 1$ and

$$R_2 \leq Z(G_{\alpha+2} \cap G_{\alpha+1}),$$

since x is an element of L.

As we also have, from 4.11(a), that (α,α') is a critical pair and there exists $\alpha'+2$ such that $d(\alpha',\alpha'+2) = 2$ and $(\alpha'+2,\alpha+2)$ is critical. Now we use the assumption that $b \geq 4$. Then we have $Z_{\alpha'+2} \leq Q_{\alpha'-2}$ and so

$$[R_2, Z_{\alpha'+2}] = 1,$$

because $R_2 \leq Z_{\alpha'-2}$. We therefore have

$$G_{\alpha+2} \cap G_{\alpha+3} = Q_{\alpha+2}Z_{\alpha'+2}, \qquad \text{(from 4.11(b))}$$

centralized by R_2. It follows that $R_2 \leq Z(G_{\alpha+2})$ and also $R_2 \leq Z(G_{\alpha-2})$ after conjugation with x in L. This contradicts the action of $Z_{\alpha'-2}$ on Z_α, as in Lemma 4.11(b), (e) and (f). We therefore have proved (4.31).

We now are finally in a position to prove the theorem, by showing that (4.31) and (4.28) contradict each other.

Because we have $Q \leq O_2(L^t) \leq Q_{\alpha+2}$ we see that

$$[Q, Z_{\alpha+2}] = 1. \tag{4.32}$$

Now we have to distinguish between two cases: $Z_{\alpha+2} \not\leq O_2(L)$ and $Z_{\alpha+2} \leq O_2(L)$.

In the first case $Q_{\alpha+1} = O_2(L)Z_{\alpha+2}$ and

$$L = \langle Z_{\alpha+2}^L \rangle O_2(L) = C_L(Q)O_2(L).$$

This shows that $O^2(L) \leq C_L(Q)$ since Q is normal in L. In particular this means $[Q, D] = 1$, which contradicts (4.28).

We therefore only have to consider the second case $Z_{\alpha+2} \leq O_2(L)$. Then we have $Z_{\alpha+2} \leq Q_\alpha$, and (4.31) and (4.20) show that we must have $b = 4$. We can now use (4.30) to see that

$$Z_{\alpha+2} \not\leq Q_{\alpha-2} = Q_{(\alpha+2)^x},$$

and L^{tx} is a normal subgroup of index 2 in $G_{\alpha-2}$. The subgroup

$$\langle (Z_{\alpha+2})^{L^{tx}} \rangle \qquad (\leq G_0)$$

contains a Sylow 3-subgroup D_2 of $G_{\alpha-2}$. Now as above, (4.32) and $Q \trianglelefteq G_0$ show that $[Q, D_2] = 1$. This contradicts (4.28) since D_2 is a G_0-conjugate of the Sylow 3-subgroup D of G_α. So the result is proved.

□

4.6.1 General Conclusions

In the general case, if $[Z_\alpha, Z_\beta] \neq 1$, the amalgam method leads to the construction of F-modules, namely for critical pairs (α, β), Z_β is an F-module of

$$\overline{X_\beta} = X_\beta / C_{X_\beta}(Z_\beta).$$

Even if $[Z_\alpha, Z_\beta] = 1$ then if V_β is defined as in Theorem 4.13, it turns out for critical pairs (α, β) that V_β is a quadratic module for $X_\beta / C_{X_\beta}(V_\beta)$. Then extra assumptions can be used to narrow down the possible groups.

4.7 Worked Examples

We give two examples of groups satisfying $\mathcal{G}$: one example for $\mathcal{H}_1$ and one example for $\mathcal{H}_2$. These are sourced from [7, pp. 299–301].

4.7.1 Worked Example Satisfying $\mathcal{G}$ and $\mathcal{H}_1$

We will consider the case $G = \mathrm{GL}_3(2)$. This is the general linear group of degree 3 over $\mathbb{F}_2$, the field of two elements. That is, all invertible 3 by 3 matrices over $\mathbb{F}_2$.

Denote

$$P_1 := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \mid a, b, c, d, e, f, g \in \mathbb{F}_2 \right\}.$$

Then P_1 is a subgroup of G as if x_1 and x_2 are defined as:

$$x_i := \begin{pmatrix} a_i & b_i & c_i \\ 0 & d_i & e_i \\ 0 & f_i & g_i \end{pmatrix}.$$

Then we have

$$x_i^{-1} = \frac{1}{a_i(d_i g_i - e_i f_i)} \begin{pmatrix} d_i g_i - e_i f_i & c_i f_i - b_i g_i & b_i e_i - c_i d_i \\ 0 & a_i g_i & -a_i e_i \\ 0 & -a_i f_i & a_i d_i \end{pmatrix},$$

where the denominator is the determinant of x_i so is non-zero (as x_i is in $\mathrm{GL}_3(2)$) and because we are working in $\mathbb{F}_2$ this is the identity element, so the determinant of x_i^{-1} is

$$\det(x_i^{-1}) = (a_i(d_i g_i - e_i f_i))^2 = (\det(x_i))^2 \neq 0.$$

Therefore x_i^{-1} is in $\mathrm{GL}_3(2)$ and in P_1. We also get:

$$\begin{aligned} x_1 \cdot x_2 &= \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & f_1 & g_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & f_2 & g_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 + c_1 f_2 & a_1 c_2 + b_1 e_2 + c_1 g_2 \\ 0 & d_1 d_2 + e_1 f_2 & d_1 e_2 + e_1 g_2 \\ 0 & f_1 d_2 + g_1 f_2 & f_1 e_2 + g_1 g_2 \end{pmatrix} \in P_1. \end{aligned}$$

We therefore have P_1 being a subgroup of G.

Now denote P_2 as follows:

$$P_2 := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \mid a, b, c, d, e, f, g \in \mathbb{F}_2 \right\}.$$

Similarly, for

$$y_i := \begin{pmatrix} a_i & b_i & c_i \\ d_i & e_i & f_i \\ 0 & 0 & g_i \end{pmatrix},$$

we have

$$y_i^{-1} = \frac{1}{g_i(a_ie_i - b_id_i)} \begin{pmatrix} e_ig_i & -b_ig_i & b_if_i - c_ie_i \\ -d_ig_i & a_ig_i & c_id_i - a_if_i \\ 0 & 0 & a_ie_i - b_id_i \end{pmatrix},$$

and again as we are working in $\mathbb{F}_2$ and

$$\det(y_i) = g_i(a_ie_i - b_id_i) \neq 0$$

we must have the determinant of y_i^{-1} as

$$\det(y_i^{-1}) = (g_i(a_ie_i - b_id_i))^2 = (\det(x_i))^2) \neq 0.$$

Therefore y_i^{-1} is in both $\mathrm{GL}_3(2)$ and P_2. Finally we have

$$\begin{aligned} y_1 \cdot y_2 &= \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ 0 & 0 & g_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ 0 & 0 & g_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1a_2 + b_1d_2 & a_1b_2 + b_1e_2 & a_1c_2 + b_1f_2 + c_1g_2 \\ d_1a_2 + e_1d_2 & d_1b_2 + e_1e_2 & d_1c_2 + e_1f_2 + f_1g_2 \\ 0 & 0 & g_1g_2 \end{pmatrix} \in P_2. \end{aligned}$$

So P_2 is also a subgroup of $\mathrm{GL}_3(2)$.

We now need to check 3 things: that G, P_1 and P_2 satisfy $\mathcal{G}$, $\mathcal{H}_1$ and G is generated by P_1 and P_2.

Now define two mappings from P_1 and P_2 to $\mathrm{SL}_2(2)$ the group of 2 by 2 matrices with entries in $\mathcal{F}_2$ and with determinant 1:

$$\varphi_1 : P_1 \longrightarrow \mathrm{SL}_2(2) \qquad \varphi_1(x_i) \mapsto \begin{pmatrix} d_i & e_i \\ f_i & g_i \end{pmatrix};$$

and

$$\varphi_2 : P_2 \longrightarrow \mathrm{SL}_2(2) \qquad \varphi_2(y_i) \mapsto \begin{pmatrix} a_i & b_i \\ d_i & e_i \end{pmatrix}.$$

Now $\varphi_1(x_i)$ and $\varphi_2(y_i)$ are members of $\mathrm{SL}_2(2)$ because:

$$\det(x_i) = a_i \cdot \det(\varphi_1(x_i)),$$

and

$$\det(y_i) = g_i \cdot \det(\varphi_2(y_i)),$$

and because x_i and y_i are members of $\mathrm{GL}_3(2)$ we must have the determinants of $\varphi_1(x_i)$ and $\varphi_2(y_i)$ non-zero and so (because we are working over $\mathbb{F}_2$) they must be equal to 1. The mappings are clearly onto and they have kernels:

$$\ker(\varphi_1) = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_2 \right\} \cong C_2 \times C_2;$$

and

$$\ker(\varphi_2) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_2 \right\} \cong C_2 \times C_2.$$

Now $\ker(\varphi_i)$ has a complement in P_i that acts faithfully on $\ker(\varphi_i)$ and so

$$P_1 \cong S_4 \cong P_2,$$

and our example satisfies $\mathcal{H}$.

Now we calculate T, the intersection of P_1 and P_2. Elements of T are of the form

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

However, T is a subgroup of $\mathrm{GL}_3(2)$ and so elements must have non-zero determinant, and therefore a, d and f are all non-zero, and so are equal to 1. We therefore have

$$T := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_2 \right\},$$

and as the order of $\mathrm{GL}_3(2)$ is 168 $(= 2^3 \cdot 3 \cdot 7)$ and the order of T is 8 $(= 2^3)$ T is a Sylow 2-subgroup of any subgroup of $\mathrm{GL}_3(2)$ which contains T. So T is a Sylow 2-subgroup of P_1 and P_2 as required. The other conditions in $\mathcal{G}$ are easy to see and so we need only now see that G is generated by P_1 and P_2.

We see this by first noting

$$|P_1P_2| = \frac{|P_1||P_2|}{|T|} = \frac{|S_4|^2}{8} = 72,$$

and then consider

$$\begin{aligned} |G : \langle P_1, P_2 \rangle| &\leq \frac{|G|}{|P_1P_2|} \\ &= \frac{168}{72} \\ &< 3. \end{aligned}$$

As the order of G is not divisible by 16 (which would be needed if the index of $\langle P_1, P_2 \rangle$ in G were 2) we have $G = \langle P_1, P_2 \rangle$. Therefore $\mathrm{GL}_3(2)$, P_1 and P_2 satisfy $\mathcal{G}$ and $\mathcal{H}_1$.

4.7.2 Worked Example Satisfying $\mathcal{G}$ and $\mathcal{H}_2$

To see $\mathcal{H}_2$ we consider the case $G = S_6$, the symmetric group of degree 6. Let

$$a := (12) \qquad \text{and} \qquad b := (12)(34)(56).$$

Now consider the centralizers of these elements:

$$P_1 := C_G(a) \qquad \text{and} \qquad P_2 := C_G(b),$$

and consider an element of P_1. Then under conjugation of x the set $\{1, 2\}$ must be invariant. This is equivalent to the set $\{3, 4, 5, 6\}$ being invariant under conjugation by x. Therefore the elements of P_1 are of the form $(12)^i c$ where $i = 0, 1$ and c is an element of the symmetric group on the elements 3, 4, 5 and 6 (denoted $\Sigma_{\{3,4,5,6\}}$). Therefore

$$P_1 = \langle a \rangle \times \Sigma_{\{3,4,5,6\}} \cong C_2 \times S_4,$$

and we get

$$O_2(P_1) = \langle a \rangle \times \langle (34)(56) \rangle \times \langle (35)(46) \rangle.$$

We also get that

$$T := O_2(P_1)\langle (34) \rangle,$$

is a Sylow 2-subgroup of P_1.

Now considering P_2 we see that for x an element of G and defining

$$\Omega := \{(12), (34), (56)\},$$

we get that x is an element of P_2 if and only if

$$\Omega^x = \Omega,$$

and hence P_2 acts on Ω. The kernel of this action is clearly

$$N := \langle (12) \rangle \times \langle (34) \rangle \times \langle (56) \rangle,$$

and we get

$$P_2/N \cong \Sigma_\Omega \cong S_3,$$

and so we see that N is in fact equal to $O_2(P_2)$. We therefore have

$$T = N\langle (35)(46) \rangle,$$

and this is a Sylow 2-subgroup of P_2.

We have P_1 and P_2 non-equal and as

$$|P_1 : T| = 3 = |P_2 : T|,$$

we get that, as required,

$$T = P_1 \cap P_2.$$

We therefore have this example satisfying all of $\mathcal{G}$ and $\mathcal{H}$ and we have only to check that G is generated by P_1 and P_2.

As in the previous example we look at the index of $\langle P_1, P_2 \rangle$ in G,

$$\begin{aligned} |G : \langle P_1, P_2 \rangle| &\leq \frac{|G|}{|P_1 P_2|} \\ &= \frac{6!}{3 \cdot 48} \\ &= 5 \end{aligned}$$

and because P_1 is not contained in A_6 we see that in fact G is equal to $\langle P_1, P_2 \rangle$ and so S_6, P_1 and P_2 satisfy both $\mathcal{G}$ and $\mathcal{H}_2$.

Chapter 5

Generalizations and Applications to Identifying Groups

The idea of an amalgam can be extended to the idea of a general amalgam over a connected partially ordered set. The definitions in this section are from [4, Chapters 28, 29].

5.1 Generalized Amalgams

Definition 5.1. Let D be a set with a partial order $\leq_D$. Then D is considered a **connected** partial order if and only if for any two elements a and b of D there is a finite sequence $x_0, x_1, \ldots, x_n$ with x_0 and x_n equal to a and b respectively and, for all i, x_i and x_{i+1} are comparable.

We can now define an amalgam based on a connected partially ordered set.

Definition 5.2. Let D be a connected partially ordered set. An **amalgam** $\mathcal{D}$ (**based on** D) is a collection $\{X_a\}_{a\in D}$ of groups along with a collection of group homomorphisms $\{\delta_{ab}\}_{\substack{a,b\in D \\ a\leq b}}$ with

$$\delta_{ab} : X_a \longrightarrow X_b$$

which satisfy the following two conditions:

- For all elements a, b and c in D with $a \leq b \leq c$ the composition $\delta_{bc}\circ\delta_{ab}$ equals δ_{ac}, so Figure 5.1 commutes; and
- For all elements a in D we have δ_{aa} equal to the identity function on X_a.

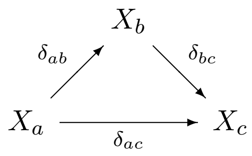

Figure 5.1: Commutative Diagram for a D-amalgam

We can denote $\mathcal{D}$ as $\mathcal{D} = \{X_a, \delta_{ab}\}$ or $\mathcal{D} = \{X_a\}$.

We can think of the amalgam as towers of groups based on equivalence classes of elements that can be compared. Unfortunately, because we are considering a connected partially ordered set rather than a totally ordered set the picture cannot be easily visualized. If we have a totally ordered set the picture is clearer with a tower of subgroups arranged by order with every subgroup connected to every subgroup ‘above’ it in the tower.

Just as with our previous notion of an amalgam we can also complete a general amalgam:

Definition 5.3. Let $\mathcal{D}$ be an amalgam defined on a connected partially ordered set D, groups X_a and group homomorphisms δ_{ab}. A **completion** of $\mathcal{D}$ is a group H with a collection of homomorphisms

$$\eta_a : X_a \longrightarrow H$$

for all a in D which satisfy:

- $\eta_a = \eta_b \circ \delta_{ab}$ for all $a \leq b$; so Figure 5.2 commutes; and
- $H = \langle \eta_a(X_a) \mid a \in D \rangle$.

We now define the general amalgam of a group X.

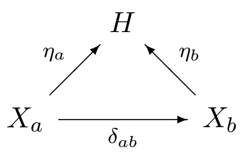

Figure 5.2: Completion rule for a general amalgam

Definition 5.4. Let X be a group, D a connected partially ordered set and $\{X_a\}_{a\in D}$ a family of subgroups of X indexed by elements of D. Let the X_a satisfy:

- X is generated by $\{X_a\}_{a\in D}$, that is
$$X = \langle X_a \mid a \in D\rangle; \text{ and}$$
- $X_a \leq X_b$ whenever $a \leq b$.

Let $\jmath_{ab} : X_a \to X_b$ for $a \leq b$ be the inclusion map. Then $\mathcal{D} = \{X_a, \jmath_{ab}\}$ is called an **X-amalgam**.

5.2 Comparing Completions and Universal Completions

We can compare completions by defining morphisms between completions:

Definition 5.5. Let $\{H, \eta_a\}$ and $\{\widetilde{H}, \widetilde{\eta_a}\}$ be two completions of an amalgam $\mathcal{D}$. A **morphism of completions** from $\{H, \eta_a\}$ to $\{\widetilde{H}, \widetilde{\eta_a}\}$ is a homomorphism
$$\Psi : H \longrightarrow \widetilde{H},$$
such that for every element a in D we have
$$\widetilde{\eta_a} = \Psi \circ \eta_a.$$

We can define epimorphisms, monomorphisms and isomorphisms of completions by considering if Ψ has the appropriate properties as a group homomorphism. We now prove that any morphism of completions is actually an epimorphism.

Lemma 5.6. *Let Ψ be a morphism of completions from H to $\widetilde{H}$. Then Ψ is surjective and $\widetilde{H}$ is contained in $\Psi(H)$.*

Proof. Let $\widetilde{h}$ be an element of $\widetilde{H}$ that is not in $\Psi(H)$. As $\widetilde{H}$ is a completion of an amalgam it is generated by the images of X_a under $\widetilde{\eta_a}$ for all a in D. Let
$$\widetilde{h} = \widetilde{\eta_{a_1}}(x_{a_1})\widetilde{\eta_{a_2}}(x_{a_2}) \cdots \widetilde{\eta_{a_n}}(x_{a_n}),$$
where each x_{a_i} is an element of X_{a_i}. Then as Ψ is a morphism of completions we can replace each $\widetilde{\eta_{a_i}}$ by $\Psi \circ \eta_{a_i}$ and so we have
$$\widetilde{h} = (\Psi \circ \eta_{a_1}(x_{a_1}))(\Psi \circ \eta_{a_2}(x_{a_2})) \cdots (\Psi \circ \eta_{a_n}(x_{a_n})).$$
However, because Ψ and the η_{a_i} are homomorphisms we get
$$\widetilde{h} = \Psi \circ [\eta_{a_1}(x_{a_1})\eta_{a_2}(x_{a_2}) \cdots \eta_{a_n}(x_{a_n})].$$

But since $\{H, \eta_a\}$ is a completion, each element $\eta_{a_i}(x_{a_i})$ is a member of H and so

$$\widetilde{h} = \Psi(h)$$

where h is a member of H. This contradicts the assumption that $\widetilde{h}$ was not in the image of H under Ψ. The second results follows from considering the images of each $\eta_{a_i}(X_{a_i})$ under Ψ, which generate $\widetilde{H}$.

□

The idea of morphisms of completions leads to a very important concept in amalgam theory - that of a completion being universal.

Definition 5.7. Let $\{H, \eta_a\}$ be a completion of an amalgam $\mathcal{D}$. We say that $\{H, \eta_a\}$ is **universal** if and only if for any given completion $\{\widetilde{H}, \widetilde{\eta_a}\}$ there is a unique morphism of completions from $\{H, \eta_a\}$ to $\{\widetilde{H}, \widetilde{\eta_a}\}$.

Proposition 5.8 ([4, 28.2]). *Let $\mathcal{D}$ be an amalgam. Then $\mathcal{D}$ has a universal completion and it is unique up to isomorphism of completions.*

Proof. We create the universal completion by creating the quotient of a free product. Let $\mathcal{D} = \{X_a, \delta_{ab}\}$ be an amalgam and let F be the free product of all the X_i. Let K be the normal subgroup of F generated by the elements

$$x^{-1}\delta_{ab}(x),$$

with x ranging over X_a and a and b ranging over all of D with $a \leq b$. Let H be the quotient F/K with the obvious mappings of X_i into H. It is clear that H is a universal completion and the definition of a universal completion along with Lemma 5.6 gives uniqueness.

□

Definition 5.9. Let $\mathcal{D}$ be an amalgam. We denote **the universal completion** by $\{\mathrm{gp}\langle\mathcal{D}\rangle, \xi_{\mathcal{D},a}\}$ where $\mathrm{gp}\langle\mathcal{D}\rangle$ is the group defined in 5.8 and

$$\xi_{\mathcal{D},a} : X_a \longrightarrow \mathrm{gp}\langle\mathcal{D}\rangle,$$

the homomorphsims.

5.3 Defining Amalgams

We make one more definition relating to types of amalgams.

Definition 5.10. Let $\mathcal{D}$ be an X-amalgam. We call $\mathcal{D}$ a **defining X-amalgam** if and only if the injections of X_i into X form a universal completion of $\mathcal{D}$.

We prove a preliminary lemma.

Lemma 5.11 ([4, 28.4]). *Let X be a group and let $\mathcal{D}$ be an X-amalgam over a connected partially ordered set D. Let $\widehat{X_a}$ and $\widehat{x}$ be the images of X_a and x in $gp\langle\mathcal{D}\rangle$ under $\xi_{\mathcal{D},a}$. Then*

(a) There is a unique homomorphism $\theta_{\mathcal{D},X}$ from $gp\langle\mathcal{D}\rangle$ into X mapping $\widehat{x}$ to X for every x in X_a for all a in D. This $\theta_{\mathcal{D},X}$ is surjective and is a morphism of completions.

(b) The mapping $\xi_{\mathcal{D},a}$ into the universal completion is an isomorphism of X_a with $\widehat{X_a}$ for all a in D. Moreover, the $\widehat{X_a}$ for a in D form a $gp\langle\mathcal{D}\rangle$-amalgam $\widehat{\mathcal{D}}$ which is isomorphic to $\mathcal{D}$.

Proof. We have part (a) following immediately from the definition of the universal completion.

For part (b) we construct the injections $\theta_{\mathcal{D},X} \circ \xi_{\mathcal{D},a}$ which inject X_a into X and the result follows.

□

We now provide equivalent definitions of a defining amalgam.

Lemma 5.12 ([4, 28.5]). *Let $\mathcal{D}$ be an X-amalgam. The following conditions are all equivalent:*

(a) The mapping $\theta_{\mathcal{D},X}$ defined in 5.11 is an isomorphism;

(b) $\mathcal{D}$ is a defining X-amalgam, that is, the injections of X_a into X form a universal completion of $\mathcal{D}$; and

(c) If $\widehat{X}$ is a completion of $\mathcal{D}$ then any morphism $\widehat{X} \to X$ of completions is an isomorphism.

Proof. Since the maps $\theta_{\mathcal{D},X}$ are from X_a into the universal completion gp$\langle\mathcal{D}\rangle$ we immediately have that (a) implies (b).

Now let (b) hold and let $\widehat{X}$ be a completion of $\mathcal{D}$. Then, because X is a universal completion of $\mathcal{D}$ (by (b)), we have a map

$$\varphi : X \longrightarrow \widehat{X}.$$

Now let ψ be a morphism of completions from $\widehat{X}$ to X as in (c). Then we can construct the morphism of completions

$$\psi \circ \varphi : X \longrightarrow X.$$

However X is a universal completion so by definition there is only one morphism of completions from X to X, that is the identity map and so φ must be injective. By Lemma 5.6 we have that φ is surjective and so it is an isomorphism and (b) implies (c).

Finally, $\theta_{\mathcal{D},X}$ is a morphism of completions and so (c) directly applies. Therefore (c) implies (a) and the equivalence is proved.

□

5.4 The Coset Geometry and Coset Complex

We can now define the general amalgam analogue of the coset graph, the coset geometry, and the topological idea of the coset complex.

Definition 5.13. We have for any X-amalgam $\mathcal{D} = \{X_a, \delta_{ab}\}$ based on a connected partially ordered set D a **coset geometry**

$$\Gamma = \Gamma(\mathcal{D}, X) = \Gamma(\mathcal{D}).$$

This consists of vertices that are cosets of the X_a's in X. Two vertices, $X_a g$ and $X_b h$ are connected by an edge if and only if $a < b$ and $X_a g \subseteq X_b h$ or $b < a$ and $X_b h \subseteq X_a g$.

We say that a vertex is of **type** a if it is of the form $X_a g$ for some g in X.

Definition 5.14. Let $\mathcal{D}$ be an X-amalgam and define the **coset complex** to be the simplicial complex whose vertices are the vertices of $\Gamma(\mathcal{D}, X)$ and whose simplices are the flags of $\Gamma(\mathcal{D}, X)$ (a flag is a subset in which any two objects are incident). We denote the coset complex by

$$\mathcal{C} = \mathcal{C}(\mathcal{D}, X) = \mathcal{C}(\mathcal{D})$$

We have the following lemma that is important in the use of the coset complex and coset geometry. They are similar results as to those already proven regarding the coset graph.

Lemma 5.15 ([4, 28.8])**.** *Let $\mathcal{D}$ be an X-amalgam based on a connected partially ordered set D. Then we have the following statements holding:*

(a) $\Gamma(\mathcal{D}, X)$ *is a connected graph;*

(b) X *acts by right translation on* $\Gamma(\mathcal{D})$ *and* $\mathcal{C}(\mathcal{D}, X)$ *transitively on objects of a given type. For every element x in X, right translation by x preserves incidence in* $\Gamma(\mathcal{D}, X)$ *and preserves simplices in* $\mathcal{C}(\mathcal{D}, X)$ *(that is, sends incident vertices to incident vertices and sends simplices to simplices);*

(c) The stabilizer of a vertex in $\Gamma(\mathcal{D}, X)$ *of type a is conjugate to X_a and the kernel of the action of X on both* $\Gamma(\mathcal{D}, X)$ *and* $\mathcal{C}(\mathcal{D}, X)$ *is the intersection of all conjugates of all X_a for a in D.*

Proof. Both (b) and (c) follow immediately from the definitions of $\Gamma(\mathcal{D}, X)$ and $\mathcal{C}(\mathcal{D}, X)$.

We see (a) by noting that the connectedness of objects in $\Gamma(\mathcal{D}, X)$ is an equivalence relation preserved by the action of X on $\Gamma(\mathcal{D}, X)$. Now pick a in D and let Γ_0 be the connected component of the coset geometry containing the vertex X_a. Then Γ_0 is preserved under the action of the stabilizer of

X_a, which is equal to X_a. As D is connected, every trivial coset X_b, for b in D is connected to X_a and so is contained in Γ_0. Then the stabilizer of Γ_0 contains the group generated by all the X_a, which is equal to X. However, X is transitive on objects of a given type and so $\Gamma_0 = \Gamma$, and hence (a) holds.

□

We now give a few definitions pertaining to the coset complex.

Definition 5.16. Let $\mathcal{C}$ and $\widehat{\mathcal{C}}$ be simplicial complexes. We say a simplicial mapping (one preserving simplices),

$$\chi : \widehat{\mathcal{C}} \longrightarrow \mathcal{C},$$

is a **local isomorphism** if for every vertex $\widehat{x}$ in $\widehat{\mathcal{C}}$ and its image $\chi(\widehat{x})$ in $\mathcal{C}$, χ induces an isomorphism of simplicial complexes between the stars of these vertices, $\mathrm{st}_{\widehat{\mathcal{C}}}(\widehat{x})$ and $\mathrm{st}_{\mathcal{C}}(\chi(\widehat{x}))$ (a star of a vertex x is a subcomplex of the simplical complex composed of all simplices of which x is a vertex).

Definition 5.17. Let $\mathcal{C}$ and $\widehat{\mathcal{C}}$ be simplicial complexes. We say a simplicial mapping (one preserving simplices)

$$\chi : \widehat{\mathcal{C}} \longrightarrow \mathcal{C}$$

is a **covering** if and only if it is surjective on the mapping of vertices and it is a local isomorphism.

We now define an important property of simplicial complexes.

Definition 5.18. Let $\mathcal{C}$ be a simplical complex. Then we say that $\mathcal{C}$ is **simply connected** if and only if it is connected (in a simplical sense) and whenever $\widehat{\mathcal{C}}$ is another connected simplical complex and χ is a covering from $\widehat{\mathcal{C}}$ to $\mathcal{C}$ then χ is an isomorphism.

5.5 Applying General Amalgams to Identifying Simple Groups

In this section we look at ways general amalgams can be used to identify groups, and in particular simple groups. We use the following key result.

Theorem 5.19 ([4, 29.1]). *Suppose we have the following conditions:*

- *G is a group and $\mathcal{D} = \{X_a\}$ is an G-amalgam;*
- *$\widehat{G}$ is a group and $\widehat{\mathcal{D}} = \{\widehat{X}_a\}$ is a defining $\widehat{G}$-amalgam; and*

- *$\mathcal{D}$ and $\widehat{\mathcal{D}}$ are based on the same connected partially ordered set D and there exists a surjective morphism φ of amalgams from $\widehat{\mathcal{D}}$ onto $\mathcal{D}$.*

  Then there exists a surjective homomorphism from $\widehat{G}$ to G.

Proof. By the definition of a morphism of amalgams the morphism φ consists of surjective group homomorphisms φ_a from $\widehat{X}_a$ to X_a with a in D. These φ_a make G a completion of $\widehat{\mathcal{D}}$ and so, because $\widehat{G}$ is a universal completion of $\widehat{\mathcal{D}}$, there exists another homomorphism χ from $\widehat{G}$ to G such that for all a in D

$$\chi|_{\widehat{X}_a} = \varphi_a.$$

We therefore have the image of $\widehat{X}$ under χ containing X_a for every a in D and, as these generate G, we must have χ an onto homomorphism, as required.

□

5.5.1 Revisiting Defining Amalgams

As Theorem 5.19 relies heavily on the fact that one of the amalgams is a defining amalgam we now look at two ways of determining if an amalgam is a defining X-amalgam. First we consider the coset complex, and link being a defining amalgam with the property of a simplical complex being simply connected. Secondly we look at the connected partially ordered set D and prove two results relating to amalgams defined on a superset of D.

Theorem 5.20 ([4, 29.2]). *Let $\mathcal{D} = \{X_a\}$ be an X-amalgam over a connected partially ordered set D. If the coset complex $\mathcal{C}(\mathcal{D}, X)$ is simply connected, then $\mathcal{D}$ is a defining X-amalgam.*

Proof. Let $\widehat{X} = \mathrm{gp}\langle\mathcal{D}\rangle$, as defined in Definition 5.9, let $\widehat{X}_a$ be the image of X_a in $\widehat{X}$ for each a in D and let $\widehat{\mathcal{D}}$ be the $\widehat{X}$-amalgam comprising of the $\widehat{X}_a$.

Let $\theta = \theta_{\mathcal{D},X}$ be the homomorphism defined in Lemma 5.11 from $\widehat{X}$ to X which is surjective. We aim to show that θ is in fact an isomorphism, and hence, from Lemma 5.21 we see that $\mathcal{D}$ is in fact a defining X-amalgam. From Lemma 5.11 we see that for each a in D, the restriction

$$\theta|_{\widehat{X}_a} : \widehat{X}_a \longrightarrow X_a,$$

is an isomorphism and so θ sends cosets of $\widehat{X}_a$ to cosets of X_i for all i in D and so induces a mapping which sends vertices of $\widehat{\mathcal{C}}(\widehat{\mathcal{D}}, \widehat{X})$ to vertices of $\mathcal{C}(\mathcal{D}, X)$. Let this mapping be denoted by Θ. It is clear that Θ preserves simplices and is surjective on vertices of $\widehat{\mathcal{C}}$. Finally, as we know that θ is a group homomorphism, it is clear that Θ preserves the action of elements of $\widehat{X}$ on $\widehat{\mathcal{C}}$ by right translation and the corresponding action of elements of X on $\mathcal{C}$.

We now show that Θ is a covering in the aim of using Definition 5.18 to prove that Θ is an isomorphism and show this implies so is θ.

Because Θ preserves the action of the groups on the corresponding coset complex it is enough to check that Θ is a local isomorphism on one vertex in each $\widehat{X}$-orbit so we simplify matters by considering only the vertices $\widehat{X}_a$.

Now let $X_b x$ be a vertex in $\mathcal{C}$ not equal to X_a. This vertex lies in the star of X_a if and only if either

(i) $a < b$ and $X_b x = X_b$; or

(ii) $a > b$ and $x \in X_a$.

These conditions translate immediately for $\widehat{\mathcal{C}}$ and so Θ is a covering due to the fact $\theta|_{\widehat{X}_a}$ is an isomorphism for all i in D.

Now we know from Lemma 5.15 that because $\mathcal{C}$ and $\widehat{\mathcal{C}}$ are coset complexes they are connected and by assumption $\mathcal{C}$ is simply connected and so Θ is an isomorphism. We now need to see simply that θ is injective to prove that $\mathcal{D}$ is defining.

Let $\widehat{x}$ be any element in the kernel of θ. We aim to show that $\widehat{x}$ is in fact the identity. As $\widehat{x}$ is in the kernel of θ its image, $\theta(\widehat{x})$ must act trivially on $\mathcal{C}$. However, Θ is an isomorphism which preserves the action of the groups on the coset complexes and so $\widehat{x}$ must act trivially on $\widehat{\mathcal{C}}$ too. Then for any a in D we must have $\widehat{x}$ being a member of the subgroup $\widehat{X}_a$ and because we know each $\theta|_{\widehat{X}_a}$ is an isomorphism we get that $\widehat{x}$ is the identity. Hence θ is an isomorphism and $\mathcal{D}$ is indeed a defining X-amalgam.

□

When paired with Theorem 5.19 this theorem provides a way of comparing groups by considering coset complexes which may be easier to analyze, particularly if the groups are of large order but the cosets are more manageable.

We now consider how a superset of the connected partially ordered set D is related to the amalgam $\mathcal{D}$.

Theorem 5.21 ([4, 29.4]). *Let D be a connected partially ordered subset of the connected partially ordered set E and let X be a group with $\mathcal{D} = \{X_a\}$ an X-amalgam defined on D. Now for each b in $E \setminus D$ define the following set, group and amalgam:*

$$\begin{aligned} D_b &:= \{a \in D \mid a \leq b\}; \\ X_b &:= \langle X_a \mid a \in D_b \rangle; \text{ and} \\ \mathcal{D}_b &:= \{X_a \mid a \in D_b\}. \end{aligned}$$

Now assume the following conditions hold:

(i) For every b in $E \backslash D$, D_b is connected and $\mathcal{D}_b$ is a defining X_b-amalgam; and

(ii) Letting $\mathcal{D}^* := \{X_b \mid b \in E\}$*, then* $\mathcal{D}^*$ *is a defining* X*-amalgam.*

Then $\mathcal{D}$ *is a defining* X*-amalgam.*

Proof. Let $\widehat{X}$ be $\mathrm{gp}\langle\mathcal{D}\rangle$, θ the map $\theta_{\mathcal{D},X}$ defined in Lemma 5.11 and $\widehat{X_a}$ the image of X_a for all a in D.

For all b in $E \setminus D$ let

$$\widehat{X_b} = \langle \widehat{X_a} \mid a \in D_b \rangle,$$

a subgroup of $\widehat{X}$. Then we can use assumption (i) to see that for every b in $E \setminus D$, $\theta|_{\widehat{X_b}}$ is an isomorphism to X_b.

Now we have proven in Lemma 5.11 (ii) that the same is true for elements of D and so the inverses of the isomorphisms $\theta|_{\widehat{X_b}}$ for b in E make $\widehat{X}$ a completion of $\mathcal{D}^*$. By assumption (ii) θ is an isomorphisms and so $\mathcal{D}$ is a defining X-amalgam as required.

□

Corollary 5.22 ([4, 29.5]). *Let* $\mathcal{E} = \{X_a, \epsilon_{ab}\}$ *be an* X*-amalgam based on a connected partially ordered set* E *and let* D *be a connected subset of* E*. Let* $\mathcal{D}$ *be the amalgam comprising of* X_a *and* ϵ_{ab} *for all* a *and* b *in* D*. Then if* $\mathcal{D}$ *is a defining* X *amalgam so is* $\mathcal{E}$*.*

Proof. This proof emulates the proof of Theorem 5.21.

□

The above results are used to identify defining amalgams. Then Theorem 5.19 with additional assumptions and specialized techniques can be used to pin down the groups the amalgams are based on.

Chapter 6

Conclusion

We now summarize the findings of this dissertation.

- An amalgam is a very natural way of thinking of a group 'sitting inside' two other groups and there is a logical idea of completing an amalgam. The idea of an amalagam can be seen to have a practical application in the identification of finite simple groups by considering the normalizer of a Sylow 2-subgroup 'sitting inside' two maximal 2-local subgroups and by using techniques such as the amalgam method to identify the groups involved in the structure.

- The coset graph Γ is a way of visualizing the structure of a group G by considering how cosets of two subgroups, P_1 and P_2, interact inside G. As we have seen this is always bipartite and there is an action of G on its vertices and edges.

- The coset graph is inextricably linked to amalgams and their structure - group-theoretic properties of G, P_1 and P_2 are linked with graph-theoretic properties of Γ - with one of the most noteworthy results being that Γ is connected if and only if G is generated by P_1 and P_2 (Theorem 3.6).

- The amalgam method relies on this link to analyze the coset graph and deduce the structure of a group given a list of assumptions $\mathcal{G}$. This is done by considering critical pairs: pairs of adjacent vertices satisfying a property equivalent to the desired conclusion.

- Both amalgams and the coset graph can be generalized leading to the idea of a defining amalgam and coset complex. A defining amalgam, in some way, defines the group on which it is based and this is linked to whether the simplicial coset complex is simply connected. We can extend the amalgam method in order to analyze simplicial complexes and thus deduce further structure of groups. This would be a very interesting research area and has been used to clear up some recognition issues in the proof of the classification of finite simple groups.

Appendix A

Schur–Zassenhaus

For self containment, we include a proof of the Schur–Zassenhaus Theorem 4.4, as given in [7, 6.2.1].

A.1 Preliminaries

We first prove Dedekind's rule, the Frattini Argument and Gaschütz's Theorem (the Schur–Zassenhaus Theorem for K an Abelian subgroup of G).

Theorem A.1 (Dedekind's Rule [1, 2.14]). *For $A, B, C \leq G$ and $B \leq A$ we have*

$$A \cap (BC) = B(A \cap C)$$

Proof. Let bg be in $B(A \cap C)$ with b in B and g in $A \cap C$. As g is in C we have bg in BC and as $B \leq A$ we also have bg in A so $B(A \cap C) \subseteq A \cap (BC)$.

Now let a be in $A \cap BC$ so $a = bc$ for some $b \in B$ and $c \in C$. Then $b^{-1}a = c$ is in A and in C as $B \leq A$ so a is in $B(A \cap C)$ and so

$$A \cap (BC) \subseteq B(A \cap C).$$

The result follows.

□

Theorem A.2 (Frattini Argument [7, 3.1.4, 3.2.7]). *Let N be a normal subgroup of G and P a Sylow p-subgroup of N. Then*

$$G = N_G(P)N.$$

Proof. G acts on the set $\Omega = \mathrm{Syl}_p(N)$ by conjugation, and the stabilizer of P is $N_G(P)$. We also have, from Sylow's Theorems that N acts transitively on Ω.

Now let g be an element of G. The transitivity of N on Ω produces an element h in N such that

$$P^g = P^h.$$

However, then we have

$$P^{gh^{-1}} = P$$

and so gh^{-1} is in the stabilizer of P, $N_G(P)$. We therefore have g begin in $N_G(P)g$ which is a subset of $N_G(P)N$ and so $G = N_G(P)N$.

□

Theorem A.3 (Gaschütz [7, 3.3.1])**.** *Let K be an Abelian normal subgroup of G such that $(|K|, |G : K|) = 1$. Then K has a complement in G and all complements of K are conjugate in G.*

Proof. First we define a relation on $\mathcal{S}$, the transversal of K in G. Let R and S be elements of $\mathcal{S}$ and define

$$R|S := \prod_{\substack{(r,s)\in R\times S \\ Kr=Ks}} (rs^{-1}) \qquad (\in K).$$

We define the relation by

$$R \sim S \iff R|S = 1.$$

This is an equivalence relation on $\mathcal{S}$ and let us denote by $\widetilde{R}$ the equivalence class which contains R.

Now

$$xR|xS = \prod_{\substack{(r,s)\in R\times S \\ Kxr=Kxs}} x(rs^{-1})x^{-1} = x(R|S)x^{-1},$$

and so if $R \sim S$ we also have $xR \sim xS$. We can therefore define an action of G on $s/\sim$ by

$$\widetilde{S}^x := \widetilde{x^{-1}S}.$$

Now let α be the automorphism of K sending l to $l^{|G/K|}$ and let

$$k := (R|S)^{-\alpha^{-1}},$$

then $\widetilde{R}^k = \widetilde{S}$ and so K acts transitively on $S/\sim$. We also have

$$R|S = 1 = kR|S \Longrightarrow k = 1$$

and so the stabilizer of $\widetilde{R}$ in K is trivial and we can apply a generalized version of Theorem A.2 to see that

$$G_{\widetilde{R}} = \{x \in G \mid xR|R = 1\},$$

is a complement of K in G and existence is proven.

Now let X be a complement of K in G and so for all $x \in X$ we have $xX = X$ and $xX|X = 1$. We therefore have $X = G_{\widetilde{X}}$ and, as K acts transitively and $(G_{\widetilde{X}})^l = G_{\widetilde{X}^l}$, we see all complements are conjugate.

□

A.2 Proof

We are now in a position to prove the main theorem.

Theorem A.4 (Schur–Zassenhaus [7, 6.2.1])**.** *Let G be a group and K a normal subgroup of G such that $(|K|, |G/K|) = 1$. Then K has a complement in G. If in addition K or G/K is soluble, then all such such complements are conjugate in G.*

Proof. Let U be a subgroup of G and N normal in G. Then we have

$$UK/K \cong U/U \cap K \qquad \text{and} \qquad (G/N)/(KN/N) \cong G/KN.$$

Therefore $U \cap K$ is a normal subgroup of U with

$$(|U \cap K|, |U/U \cap K|) = 1,$$

and KN/N is a normal subgroup of G/N with

$$(|KN/N|, |G/KN|) = 1,$$

and so our hypothesis is inherited through taking subgroups and quotients of G. If we also have K or G/K soluble then this property is also inherited.

We now induct on the order of G to prove existence of the complement. We can therefore assume all groups of order less than G which satisfy the hypothesis have the necessary complement. We can also assume that $1 \neq K < G$.

Now let p be a prime dividing the order of K and let P be a Sylow p-subgroup of K. denote

$$U := N_G(P),$$

and first assume that U is not equal to G. Then by induction $U \cap K$ has a complement in U. Theorem A.2 give us that

$$G = KU = K(U \cap K)H = KH.$$

We therefore have H another complement of K in G since $H \cap K$ is equal to $H \cap (U \cap K)$ which is equal to 1.

Now assume that $U = G$. Then P and also

$$N := Z(P), \quad (\neq 1)$$

is a normal subgroup of G. Let $\overline{G} := G/N$, then by induction there exists $N \leq V \leq G$ such that $\overline{V}$ is a complement of $\overline{K}$ in $\overline{G}$. However, then

$$V \cap K = N \qquad \text{and} \qquad G = KV$$

and so a complement of N in V is in fact also a complement of K in G. Now if V is not equal to G then by induction such a complement exists. If V is equal to G then $\overline{K}$ is trivial and so K is Abelian and we can apply Theorem A.3 to obtain the desired complement.

Now let K or G/K be soluble and we induct on the order of G to prove that all complements are conjugate. Let H_1 and H_2 be two complements of K in G and let N be a minimal normal subgroup of G that is contained in K. Once again set $\overline{G} := G/N$ and so $\overline{H_1}$ and $\overline{H_2}$ are complements of $\overline{K}$ in $\overline{G}$. By induction there exists a g in G such that

$$H_1N = (H_2N)^g = H_2{}^gN,$$

and so H_1 and $H_2{}^g$ are complements of N in H_1N. If $N \neq K$ then $H_1N \neq G$ and so by induction H_1 and $H_2{}^g$ are conjugate in H_1N and so H_1 and H_2 are conjugate in G.

Now assume that $N = K$. If K is soluble then N is a soluble minimal normal subgroup and so must be Abelian (in fact, Elementary Abelian) and so H_1 and H_2 are conjugate by Theorem A.3.

Finally, assume K is not soluble, so $\overline{G}$ is soluble. Then there must be a normal subgroup $\overline{A}$ in $\overline{G}$ with $K \leq A \trianglelefteq G$ and $\overline{G}/\overline{A}$ a nontrival p-group. By Dedekind's rule A.1 shows that $H_1 \cap A$ and $H_2 \cap A$ are complements of K in A and so by induction they are conjugate in A.

We therefore have (after conjugation)

$$H_1 \cap A = H_2 \cap A := D \trianglelefteq \langle H_1, H_2 \rangle.$$

Now since $H_1/D \cong G/A \cong H_2/D$ there must exist Sylow p-subgroups P_1 of H_1 and P_2 of H_2 with

$$H_i = DP_i.$$

Also, because $(|K|, |H|) = 1$ we see that P_1 and P_2 are Sylow p-subgroups of $N_G(D)$ and so by Sylow's Theorems there must exist g in $N_G(D)$ with $P_2^g = P_1$ and so

$$H_2{}^g = D^gP_2{}^g = DP_1 = H_1,$$

and the theorem is proved.

□

Appendix B

Coprime Action on G

We now give some results that facilitate proofs concerning Q_α and Z_α.

Lemma B.1 ([7, 8.2.1, 8.2.2(a), 8.2.7]). *Suppose A acts coprimely on G. Let N be an A-invariant subgroup of G with the action of A on N also coprime. Let g be in G and be such that $(Ng)^A = Ng$. Then there exists c in $C_G(A)$ such that*

$$Ng = Nc.$$

It follows that

$$C_{G/N}(A) = C_G(A)N/N.$$

It also follows that

$$G = [G, A]C_G(A).$$

Proof. As $N^A = N$ and $(Ng)^A = Ng$ we must have $g^a g^{-1}$ in N for all a in A.

In the semidirect product AG we get $a^{-1}gag^{-1}$ in N and

$$A^{g^{-1}} \leq AN.$$

Hence A and $A^{g^{-1}}$ are complements of N in AN and, using the fact A is acting coprimely on G along with 4.4, they are conjugate in AN.

Thus there exists an h in N such that $A^h = A^{g^{-1}}$. For $c := hg$ this gives

$$c \in N_{AG}(A) \cap Ng,$$

and $[A, c] \leq A \cap G = 1$.

The second result follows immediately and the final result follows with $N := [G, A]$.

□

Corollary B.2 ([7, 8.2.2(b)]). *Let N be an A-invariant normal subgroup of G. Suppose that the action of A on N is coprime. Then if A acts trivially on N and G/N, then A acts trivially on G.*

Proof. This follows as a consequence of Lemma B.1 where we see that

$$C_{G/N}(A) = C_G(A)N/N.$$

□

Lemma B.3 ([7, 8.4.2]). *Let the action of a group A on a group G be coprime. Then*

$$G = C_G(A) \times [G, A]$$

Proof. By Lemma B.1 we need only show that the intersection $C_G(A) \cap [G, A] = 1$. To do this we consider the following endomorphism

$$\varphi : G \longrightarrow G \qquad g \longmapsto \prod_{x \in A} g^x.$$

Now consider a commutator $g = [h, a]$ in $[G, A]$. We have

$$\varphi(g) = \varphi(h^a)\varphi(h^{-1}) = \left(\prod_{x \in A} h^{ax}\right)\left(\prod_{x \in A} h^{-x}\right) = 1,$$

so $[G, A] \leq \ker(\varphi)$.

Conversely, for g in $C_G(A)$ we get

$$\varphi(g) = g^{|A|}$$

and

$$g^{|A|} = 1 \Longleftrightarrow g = 1,$$

since $(|A|, |G|) = 1$ hence $g = 1$ for g in $C_G(A) \cap [G, A]$.

□

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