On the Geometry of Interaction for Classical Logic

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EPSRC project

“Semantics of classical proofs”, involving (alphabetically)

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Hyland,
Pym,
Robinson,
Urban.

The non-determinism of cut-reduction

The proof in the middle (attributed to Lafont) reduces to both Φ_1 and Φ_2:

$$
\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma\vdash\Delta\end{array}
\quad\succcurlyeq\quad
\dfrac{\dfrac{\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma\vdash\Delta\end{array}}{\Gamma\vdash A,\Delta}\,\mathsf{WR} \qquad \dfrac{\begin{array}{c}\Phi_2\\ \vdots\\ \Gamma\vdash\Delta\end{array}}{\Gamma, A\vdash\Delta}\,\mathsf{WL}}{\Gamma\vdash\Delta}\,\mathsf{Cut}
\quad\preccurlyeq\quad
\begin{array}{c}\Phi_2\\ \vdots\\ \Gamma\vdash\Delta.\end{array}
$$

Therefore, models that preserve meaning along cut reductions are trivial.

Flawed models

- CCC's with a dualizing object, i.e. an object $\bot$ such that the map below has an inverse for every object A.

$$A \longrightarrow ((A \to \bot) \to \bot)$$

  - Problem: such categories are boolean lattices.

- Translations into classical natural deduction.
  - Problem: admit only the left reduction in Lafont's example (call-by-value) or the right one (call-by-name).

Overview of this talk

1. Background: categorical models of MLL a'la Cockett & Seely.

2. Symmetric (co)monoids for modelling W and C; *order-enrichment* $\leq$ such that

$$\Phi \preccurlyeq \Psi \implies \mathbf{C}[\![\Phi]\!] \leq \mathbf{C}[\![\Psi]\!]$$

   (inequational soundness w.r.t. cut reduction). Examples include

   - without $\neg$: $(\mathbf{Rel}, \uplus)$, distributive lattices; free model of proof nets;
   - with $\neg$: $(\mathbf{Rel}, \times)$, boolean lattices; free model of proof nets.

3. Background: GoI for MIX categories (Cockett & Seely).

4. Extension of GoI to W and C.

Models of MLL

Models of MLL

Definition 1. [Cockett/Seely] *Symmetric linearly distributive category:*

- *Symmetric monoidal product* $\otimes$ *(for modelling* $\wedge$ *and left comma).*
- *Symmetric monoidal product* $\oplus$ *(for modelling* $\vee$ *and right comma).*
- *Natural transformation satisfying some coherence conditions:*

$$\delta : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C.$$

Modelling Cut with δ

$$\dfrac{\begin{matrix}\Phi \\ \vdots \\ \Gamma \vdash A, \Delta\end{matrix} \qquad \begin{matrix}\Psi \\ \vdots \\ \Gamma', A \vdash \Delta'\end{matrix}}{\Gamma', \Gamma \vdash \Delta', \Delta}\ \mathsf{Cut}$$

is interpreted as

$$\Gamma' \otimes \Gamma \xrightarrow{\Gamma' \otimes [\![\Phi]\!]} \Gamma' \otimes (A \oplus \Delta) \xrightarrow{\delta} (\Gamma' \otimes A) \oplus \Delta \xrightarrow{[\![\Psi]\!] \oplus \Delta} \Delta' \oplus \Delta.$$

Adding negation

To model negation, every object needs a *complement*:

Definition 2. *A complement of an object A is an object $A^{\perp}$ with maps*

$$\eta_A : \top \to (A^{\perp} \oplus A) \qquad\qquad \varepsilon_A : (A \otimes A^{\perp}) \to \perp.$$

satisfying two equations.

Fact: every object has (up to isomorphism) at most one complement.

Definition 3. *An SLDC with negation is an SLDC in which every object has a complement.*

Free SLDC generated by proof nets

Theorem 1. [Cockett/Seely] *Suppose that*

- $\mathcal{C}$ *is a polygraph,*
- $Net(\mathcal{C})$ *is the polygraph of proof nets generated by* $\mathcal{C}$*, and*
- E *be set of equations between proof nets with the same inputs and outputs.*

Then $Net(\mathcal{C})$ *modulo* E *and certain equations including cut elimination forms the free SLDC generated by* $\mathcal{C}$ *and* E*.*

Also works in the presence of negation.

Adding weakening and contraction

Modelling weakening and contraction

- A type-indexed family of symmetric monoids

$$A \oplus A \xrightarrow{\blacktriangledown_A} A \xleftarrow{[]_A} \bot$$

satisfying the evident coherence conditions.

- A type-indexed family of symmetric co-monoids

$$A \otimes A \xleftarrow{\blacktriangle_A} A \xrightarrow{\langle\rangle_A} \top$$

satisfying the evident coherence conditions.

Example: associativity

The associativity law of the monoids corresponds to

$$\dfrac{\dfrac{\begin{array}{c}\Phi \\ \vdots \\ \Gamma \vdash \Delta_1, (A, A), A, \Delta_2\end{array}}{\Gamma \vdash \Delta_1, A, A, \Delta_2}\,\mathsf{CR}}{\Gamma \vdash \Delta_1, A, \Delta_2}\,\mathsf{CR} \quad = \quad \dfrac{\dfrac{\begin{array}{c}\Phi \\ \vdots \\ \Gamma \vdash \Delta_1, A, (A, A), \Delta_2\end{array}}{\Gamma \vdash \Delta_1, A, A, \Delta_2}\,\mathsf{CR}}{\Gamma \vdash \Delta_1, A, \Delta_2}\,\mathsf{CR}.$$

Models of the classical sequent calculus

Definition 4. *A Dummett category is partial-order enriched symmetric linearly distributive category with symmetric monoids and comonoids such that*

1. *$\otimes$, and $\oplus$ are monotonic in both arguments;*
2. *parametric versions of the laws below hold.*

to model cut reductions involving C	*to model cut reductions involving W*
$f \circ \blacktriangledown \;\leq\; \blacktriangledown \circ (f \oplus f)$	$f \circ [] \;\leq\; []$
$\blacktriangle \circ f \;\leq\; (f \otimes f) \circ \blacktriangle$	$\langle\rangle \circ f \;\leq\; \langle\rangle$

A classical category is a Dummett category with $A^{\perp} \otimes A \longrightarrow \perp$ and $\top \longrightarrow A \oplus A^{\perp}$.

Examples of Dummett categories

Category	$f \leq g$	negation?	$\otimes = \oplus$?
$(\mathbf{Rel}, \times)$	$f \subseteq g$	yes	yes
boolean lattice $\mathbf{B}$	trivial	yes	no
free model built from nets	$f \preccurlyeq g$	optional	no
$(\mathbf{Rel}, \uplus)$	$f \supseteq g$	no	yes
$\mathrm{GoI}(\mathbf{Rel}, \uplus)$	see later	yes	yes

Digression: MIX categories

Definition 5. [Cockett/Seely] *A MIX category is a SLDC together with a morphism*

$$m : \bot \longrightarrow \top$$

such that the two evident morphisms

$$A \otimes B \longrightarrow A \oplus B$$

agree.

Classical logic and MIX

Theorem 2. [Hasegawa/Führmann] *Every SLDC with a monoid*

$$\top \oplus \top \xrightarrow{\blacktriangledown} \top \xleftarrow{[]} \bot$$

is MIX with $m = []$*.*

Corollary 1. *Every Dummett category is MIX.*

Remark 1. *There is also a purely proof-theoretic argument showing that the classical sequent calculus should be MIX under modest assumptions.*

Hom-semilattices

Definition 6. *For two morphisms $f, g : A \longrightarrow B$ of a Dummett category, $f * g : A \longrightarrow B$ is defined as follows:*

$$f * g \quad = \quad A \xrightarrow{\blacktriangle} A \otimes A \begin{array}{c} \xrightarrow{\text{mix}} A \oplus A \xrightarrow{f \oplus g} \\ \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\text{mix}} \end{array} B \oplus B \xrightarrow{\blacktriangledown} B.$$

Theorem 3. *On every hom-set, $*$ forms a semilattice (not generally with a unit).*

Equational axiomatization of Dummett categories

Proposition 1. *The order $\leq$ is the one induced by the semilattice $*$.*

The *conditional* equalities stating the monotonicity of $\circ$, $\otimes$, and $\oplus$ can be replaced by unconditional equalities. So:

Theorem 4. *Dummett categories can be axiomatized as SLDCs with symmetric monoids and symmetric comonoids subject to a number of* unconditional *equalities.*

Free classical category generated by nets

Theorem 5. *Suppose that*

- $\mathcal{C}$ *is a polygraph containing edges for* W *and* C*,*
- $Net(\mathcal{C})$ *is the polygraph of nets generated by* $\mathcal{C}$*, and*
- E *be set of equations between nets with the same inputs and outputs.*

Then $Net(\mathcal{C})$ *modulo* E *and certain (in)equalities including cut elimination forms the free classical category generated by* $\mathcal{C}$ *and* E*.*

Proof. A special case of the free construction by Cockett & Seely, because of the previous theorem. □

GoI

Traced objects

Definition 7. *A trace for an object U of an SLDC $\mathbf{C}$ is a family of functions*

$$tr_U^{AB} : \mathbf{C}(U \otimes A, U \oplus B) \longrightarrow \mathbf{C}(A, B)$$

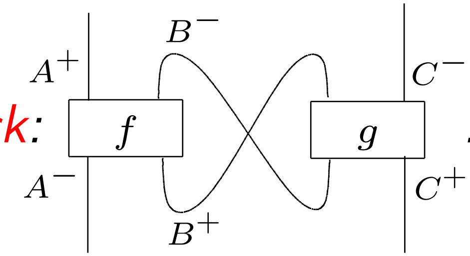
satisfying axioms called "Yanking", "Tightening", and "Superposing".

Trace and compactness

- There is a notion of compatibility between a trace on U and a trace on V.
- The GoI construction by Cockett & Seely requires a set $\mathcal{U}$ of pairwise compatible traces objects.
- If $\mathcal{U}$ contains every object, then the MIX category is compact—that is, it is essentially a symmetric monoidal category. The trace is the usual one.

The GoI construction

Definition 8. *Given a traced symmetric monoidal category* $\mathbf{C}$*, the category* $\mathrm{GoI}(\mathbf{C})$ *is defined as follows:*

- *Objects are pairs* (A^+, A^-) *of objects of* $\mathbf{C}$*;*
- *A morphism* $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ *of* $\mathrm{GoI}(\mathbf{C})$ *is a morphism* $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ *of* $\mathbf{C}$*;*
- *Composition is defined by symmetric feedback:* 

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The category $\mathrm{GoI}(\mathbf{C})$

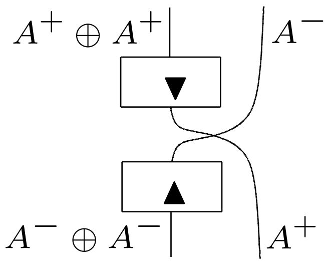
Theorem 6. [Abramsky/Joyal/Street/Verity] $\mathrm{GoI}(\mathbf{C})$ *is a compact closed category (= a SLDC with negation such that* $\otimes = \oplus$*,* $\bot = \top$*, and* $\delta = \alpha$*.).*

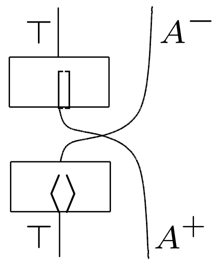
The extended GoI construction

Theorem 7. [Hasegawa] *If* $\mathbf{C}$ *is a traced Dummett category, then* $\mathrm{GoI}(\mathbf{C})$ *is a classical category.*

(Generalized version of theorem in our LICS paper.)

Proof. The order $\leq$ is inherited from $\mathbf{C}$. The monoids are defined as follows:

$\blacktriangledown_{(A^+,A^-)} =$ 

$[]_{(A^+,A^-)} =$ 

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Dually for the comonoids. (Checking the inequalities is hard.) □

Ongoing work

- More non-compact models. Games? (Pym/Ritter.)
- Extension to predicate logic (McKinley)
- More general models (but without order-enrichment) covering

$$\mathrm{id}_{A\otimes B} \neq \mathrm{id}_A \otimes \mathrm{id}_B \qquad \mathrm{id}_{A\oplus B} \neq \mathrm{id}_A \oplus \mathrm{id}_B$$

(Bellin/Hyland/Robinson/Urban).