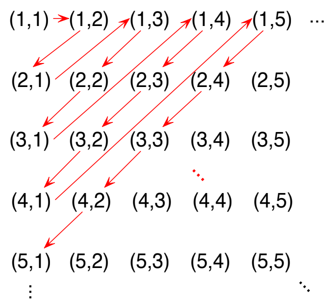
Enumerability & diagonalization

Cantor's Zig-Zag

Pairs of integers: Cantor's Zig-Zag.



$f(1) = (1,1), f(2) = (1,2), f(3) = (2,1), f(4) = (1,3), f(5) = (2,2), \ldots$

Exercise

Show the following statements.

1. Every finite set is enumerable.
2. If a **non-empty** set A is enumerable, then it is enumerable by a **total** function. Why does this not work for the empty set?

Code numbers

Definition. Given an enumeration f of a set A, a **code numbers** of an element a of A is a number n such that $f(n) = a$.
Examples:

- The code number of $(1,3)$ with respect to Cantor's Zig-Zag is 4.
- For the enumeration $2, 2, 4, 4, 6, 6, \ldots$ of the even numbers, the code numbers of 4 are 2 and 3.

Encodings

Definition. An **encoding** of a set A is a total injective function $c : A \to N$ into the natural numbers. For a in A, the number $c(a)$ is called the **code** of a.

Proposition. A set A has an encoding if and only if it is enumerable.

The proof of this proposition follows on the next two slides.

From encoding to enumeration

Let $c : A \to N$ be an encoding. Then an enumeration $f : N \to A$ is given by

$$f(n) = \begin{cases} a & \text{if } n = c(a) \\ \text{undefined} & \text{otherwise} \end{cases}$$

f is well-defined (i.e. there is only one a for every n) because c is injective.

From enumeration to encoding

Let $f : N \to A$ be an enumeration. An encoding $c : A \to N$ is given by

$$c(a) = \text{some } n \text{ such that } f(n) \text{ is equal to } a.$$

Because we need the function c to be an encoding, it must be total and injective. It is total because f is surjective. It is injective because f cannot send some n to two different values a and a'.

“Enumerable” vs. “equinumerous”

Proposition. Every enumerable set A is either finite or equinumerous with N.

Proof. Let A be a set which is enumerable but not finite. Let $c : A \to N$ be an encoding of A. We define a bijection $b : N \to A$ by

$b(1)$ = the element of A with the smallest code

$b(2)$ = the element of A with the 2nd smallest code

$b(3)$ = the element of A with the 3rd smallest code

$\vdots$

"Enumerable" vs. "equinumerous"

Proposition. A set A is enumerable **if and only if** it is finite or equinumerous with N.

Proof. The left-to-right direction is the result on the previous slide. For the right-to-left direction, suppose first that A is equinumerous with N. Then there is a bijection $b : N \to A$. In particular, A is the range of b, so A is enumerable. Second, suppose that A is finite. Then by an earlier exercise, A is enumerable.

Encoding pairs of integers

As we have seen, Cantor's Zig-Zag provides an encoding of pairs of natural numbers.

Here is an alternative encoding:

$$c_{N \times N}(m, n) = p^m \cdot q^n$$

where p and q are different primes. The total function $c_{N \times N}$ is injective owing to the **uniqueness of prime decomposition**.

Encoding other kinds of pairs

Proposition. If A and B are enumerable sets, then so is the set $A \times B$ of pairs.

Proof. Let $c_A : A \to N$ and $c_B : B \to N$ be encodings of A and B. An encoding $c_{A \times B} : A \times B \to N$ is given by

$$c_{A \times B}(a, b) = c_{N \times N}(c_A(a), c_B(b)),$$

where $c_{N \times N}$ is some encoding of pairs of integers. ($c_{A \times B}$ is injective because c_A, c_B, and $c_{N \times N}$ are.)

Exercises

Show the following statements:

1. If A, B, and C are enumerable sets, then the set of triples

   $$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

   is enumerable.
2. If $A_1, A_2, \ldots, A_k$ are enumerable sets, then the set of k-tuples $A_1 \times A_2 \times \cdots \times A_k$ is enumerable.

Exercises

Show the following statements:

1. If A is enumerable and there is a surjective function $A \to B$, then B is enumerable.
2. If B is enumerable and there is a total injective function $A \to B$, then A is enumerable.

Remark: these two statements are useful for some of the following exercises.

Exercises

Show that the following sets are enumerable:

1. The set Q^+ of positive rational numbers.
2. The set Q of all rational numbers.
3. The set $A \cup B$ for enumerable sets A and B.
4. The set A^* of strings over an enumerable alphabet A.

Exercises

Show that the following statements:

1. The set $P_{fin}(N)$ of **finite subsets** of N is enumerable.
2. The set $P_{fin}(A)$ of finite subsets of an arbitrary enumerable set A is enumerable.