Chapter 4

Declarative Interpretation
Outline

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models
What is an Interpretation?

direct(frankfurt,san_francisco).
direct(frankfurt,chicago).
direct(san_francisco,honolulu).
direct(honolulu,maui).

collection(X, Y) :- direct(X, Y).
collection(X, Y) :- direct(X, Z), collection(Z, Y).

\[ D = \{FRA, DRS, ORD, SFO, \ldots\} \]
\[ frankfurt_j = FRA, \,\text{chicago}_j = ORD, \,\text{san_francisco}_j = SFO, \ldots \]
\[ \text{direct}_i = \{(FRA, SFO), (FRA, ORD), \ldots\} \]
\[ \text{connection}_i = \{(FRA, SFO), (FRA, ORD), (FRA, HNL), \ldots\} \]
What is an Interpretation?

\[
\begin{align*}
\text{add}(X,0,X). \\
\text{add}(X,\text{s}(Y),\text{s}(Z)) & : \text{- add}(X,Y,Z).
\end{align*}
\]

\[
D = \mathbb{N}
\]

\[
0_j = 0
\]

\[
s_j : \mathbb{N} \rightarrow \mathbb{N} \text{ such that } s_j(n) = n + 1
\]

\[
\text{add}_j = \{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 2), \ldots\}
\]
Another Example

\[\text{add}(X, 0, X).\]
\[\text{add}(X, s(Y), s(Z)) \leftarrow \text{add}(X, Y, Z).\]

\[D = \{0, s(0), s(s(0)), \ldots\}\]
\[0_J = 0\]
\[s_J : D \rightarrow D \text{ such that } s_J(t) = s(t)\]
\[\text{add}_J = \{(0, 0, 0), (s(0), 0, s(0)), (0, s(0), s(0)), (s(0), s(0), s(s(0))), \ldots\}\]

(This will be called a “Herbrand model”.)
Algebras

$V$ set of variables, $F$ ranked alphabet of function symbols: An algebra $J$ for $F$ (or pre-interpretation for $F$) consists of:

1. domain $\iff$ non-empty set $D$
2. assignment of a mapping
   \[ f_j : D^n \rightarrow D \]
   to every $f \in F^{(n)}$ with $n \geq 0$

State $\sigma$ over $D$ $\iff$ mapping $\sigma : V \rightarrow D$

Extension of $\sigma$ to $TU_{F,V}$ $\iff$ $\sigma : TU_{F,V} \rightarrow D$ such that for every $f \in F^{(n)}$
\[ \sigma(f(t_1, ..., t_n)) = f_j(\sigma(t_1), ..., \sigma(t_n)) \]
Interpretations

$F$ ranked alphabet of function symbols, $\Pi$ ranked alphabet of predicate symbols:

An interpretation $I$ for $F$ and $\Pi$ consists of:

1. algebra $J$ for $F$ (with domain $D$)
2. assignment of a relation

$$p_i \subseteq D \times \ldots \times D$$

$n$ times

... to every $p \in \Pi^{(n)}$ with $n \geq 0$
Herbrand Universes and Bases

Recall  \( TU_{F,V} \leftrightarrow \) term universe over function symbols \( F \), variables \( V \)

\( TB_{\Pi,F,V} \leftrightarrow \) term base (i.e., all atoms) over predicate symbols \( \Pi \) and \( F \), \( V \)

- Herbrand universe \( HU_F \leftrightarrow TU_{F,\emptyset} \)
- Herbrand base \( HB_{\Pi,F} \leftrightarrow TB_{\Pi,F,\emptyset} \)
Interpretations (Example)

Let $P_{add}$ “add-program”.

$l_1, l_2, l_3, l_4, l_5,$ and $l_6$ are interpretations for $\{s, 0\}$ and $\{add\}$:

$l_1$: $D_{l_1} = \mathbb{N}$, $0_{l_1} = 0$, $s_{l_1}(n) = n + 1$ for each $n \in \mathbb{N}$, $add_{l_1} = \{(m, n, m + n) \mid m, n \in \mathbb{N}\}$

$l_2$: $D_{l_2} = \mathbb{N}$, $0_{l_2} = 0$, $s_{l_2}(n) = n + 1$ for each $n \in \mathbb{N}$, $add_{l_2} = \{(m, n, m \ast n) \mid m, n \in \mathbb{N}\}$

$l_3$: $D_{l_3} = HU_{\{s, 0\}}$, $0_{l_3} = 0$, $s_{l_3}(t) = s(t)$ for each $t \in HU_{\{s, 0\}}$,

$add_{l_3} = \{(s^m(0), s^n(0), s^{m+n}(0)) \mid m, n \in \mathbb{N}\}$

$l_4$: $D_{l_4} = HU_{\{s, 0\}}$, $0_{l_4} = 0$, $s_{l_4}(t) = s(t)$ for each $t \in HU_{\{s, 0\}}$, $add_{l_4} = \emptyset$

$l_5$: $D_{l_5} = HU_{\{s, 0\}}$, $0_{l_5} = 0$, $s_{l_5}(t) = s(t)$ for each $t \in HU_{\{s, 0\}}$, $add_{l_5} = (HU_{\{s, 0\}})^3$

$l_6$: $D_{l_6} = \{0, 1\}$, $0_{l_6} = 0$, $s_{l_6}(n) = n$ for each $n \in \{0, 1\}$, $add_{l_6} = \{(m, n, m) \mid m, n \in \{0, 1\}\}$
Logical Truth (I)

\[ E \text{ expression} :\iff E \text{ atom, query, clause, or resultant} \]

\[ E \text{ expression}, \; l \text{ interpretation}, \; \sigma \text{ state:} \]

\[ E \text{ true in } l \text{ under } \sigma \text{, written: } l \models_\sigma E \]

\[ \iff \]

by case analysis on \( E \):

- \( l \models_\sigma p(t_1, \ldots, t_n) \iff (\sigma(t_1), \ldots, \sigma(t_n)) \in p_i \)
- \( l \models_\sigma A_1, \ldots, A_n \iff l \models_\sigma A_i \text{ for every } i = 1, \ldots, n \)
- \( l \models_\sigma A \leftarrow B :\iff \text{if } l \models_\sigma B \text{ then } l \models_\sigma A \)
- \( l \models_\sigma A \leftarrow B :\iff \text{if } l \models_\sigma B \text{ then } l \models_\sigma A \)
Logical Truth (II)

$E$ expression, $I$ interpretation:
Let $x_1, \ldots, x_k$ be the variables occuring in $E$.

- $\forall x_1, \ldots, \forall x_k E$ universal closure of $E$ (abbreviated $\forall E$)
- $\exists x_1, \ldots, \exists x_k E$ existential closure of $E$ (abbreviated $\exists E$)
- $I \models \forall E \iff I \models_\sigma E$ for every state $\sigma$
- $I \models \exists E \iff I \models_\sigma E$ for some state $\sigma$
- $E$ true in $I$ (or: $I$ model of $E$), written: $I \models E \iff I \models \forall E$
Logical Truth (III)

$S, T$ sets of expressions, $I$ interpretation:

- $I$ model of $S$, written: $I \models S \iff I \models E$ for every $E \in S$
- $T$ semantic (or: logical) consequence of $S$, written $S \models T$
  $\iff$ every model of $S$ is a model of $T$

$P$ program, $Q_0$ query, $\theta$ substitution:

- $\theta \models_{\text{var}(Q_0)}$ correct answer substitution of $Q_0$ \iff $P \models Q_0 \theta$
- $Q_0 \theta$ correct instance of $Q_0$ \iff $P \models Q_0 \theta$
Models (Example)

Let $P_{\text{add}}$ “add-program” and let $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, and $I_6$ be the interpretations from slide 9.

- $I_1 \models P_{\text{add}}$ (since $I_1 \models_\sigma c$ for every clause $c \in P_{\text{add}}$ and state $\sigma : V \rightarrow \mathbb{N}$:
  (i) $(\sigma(x), \sigma(0), \sigma(x)) \in \text{add}_{I_1}$ and
  (ii) if $(\sigma(x), \sigma(y), \sigma(z)) \in \text{add}_{I_1}$ then $(\sigma(x), \sigma(y)+1, \sigma(z)+1) \in \text{add}_{I_1}$)

- $I_2 \not\models P_{\text{add}}$ (e.g. let $\sigma(x) = 1$, then $I_2 \not\models_\sigma \text{add}(x, 0, x)$ since $(\sigma(x), \sigma(0), \sigma(x)) = (1, 0, 1) \notin \text{add}_{I_2}$)

- $I_3 \models P_{\text{add}}$ (like for $I_1$; we call $I_3$ a (least) Herbrand model)

- $I_4 \not\models P_{\text{add}}$ (e.g. let $\sigma(x) = s(0)$, then $I_4 \not\models_\sigma \text{add}(x, 0, x)$ since $(\sigma(x), \sigma(0), \sigma(x)) = (s(0), 0, s(0)) \notin \text{add}_{I_4}$)

- $I_5 \models P_{\text{add}}$ (like for $I_1$; we call $I_5$ a Herbrand model)

- $I_6 \models P_{\text{add}}$ (like for $I_1$)
Semantic Consequences (Example)

Let $P_{\text{add}}$ “add-program”.

- $P_{\text{add}} \models add(x, 0, x)$
  (for every interpretation $I$ : if $I \models P_{\text{add}}$ then $I \models add(x, 0, x)$, since $add(x, 0, x) \in P_{\text{add}}$)

- $P_{\text{add}} \models add(x, s(0), s(x))$
  (for every interpretation $I$ : if $I \models P_{\text{add}}$ then $I \models add(x, 0, x)$
  and $I \models add(x, s(0), s(x)) \leftarrow add(x, 0, x)$ (instance of clause), thus $I \models add(x, s(0), s(x))$)

- $P_{\text{add}} \not\models add(0, x, x)$
  (consider interpretation $I_6$ from slide 9 with $I_6 \models P_{\text{add}}$:
  $I_6 \not\models add(0, x, x)$, since e.g. $I_6 \not\models add(0, x, x)$ for $\sigma(x) = 1$
  since $(\sigma(0), \sigma(x), \sigma(x)) = (0, 1, 1) \notin add_{I_6}$)
Towards Soundness of SLD-Resolution (I)

Lemma 4.3 (i)

Let \( Q \rightarrow Q' \) be an SLD-derivation step and \( Q\theta \leftarrow Q' \) the resultant associated with it.

Then \( c \models Q\theta \leftarrow Q' \)

Proof.

Let \( Q = \mathbf{A}, B, C \) with selected atom \( B \). Let \( H \leftarrow B \) be the input clause and \( Q' = (A, B, C)\theta \).

Then

\[
\begin{align*}
\cmodels c & \models H \leftarrow B \\
\text{implies} & \cmodels c \models H\theta \leftarrow B\theta \quad \text{(instance)} \\
\text{implies} & \cmodels c \models B\theta \leftarrow B\theta \quad \text{(}\theta\text{ unifier)} \\
\text{implies} & \cmodels c \models (A, B, C)\theta \leftarrow (A, B, C)\theta \quad \text{("context" unchanged)}
\end{align*}
\]
Towards Soundness of SLD-Resolution (II)

Lemma 4.3 (ii)

Let $\xi$ be an SLD-derivation of $P \cup \{Q_0\}$. For $i \geq 0$ let $R_i$ be the resultant of level $i$ of $\xi$.

Then $P \models R_i$

Proof.

Let $\xi = Q_0 \overrightarrow{\theta_1} Q_1 \overrightarrow{\theta_{n+1}} Q_n \overrightarrow{\theta_{n+1}} \ldots$ Induction on $i \geq 0$:

$i = 0$: $R_0 = Q_0 \leftarrow Q_0 = \text{"true"} \text{, thus } P \models R_0$

$i = 1$: $R_1 = Q_0 \theta_1 \leftarrow Q_1$; by Lemma 4.3 (i): $P \models R_1$

$i \Rightarrow i + 1$: $R_{i+1} = Q_0 \theta_1 \ldots \theta_{i+1} \leftarrow Q_{i+1}$ is a semantic consequence of resultant $Q_i \theta_{i+1} \leftarrow Q_{i+1}$ associated with $(i + 1)$-st derivation step and $R_i \theta_{i+1} = Q_0 \theta_1 \ldots \theta_{i+1} \leftarrow Q_i \theta_{i+1}$, thus by Lemma 4.3 (i) and induction hypothesis: $P \models R_{i+1}$
Soundness of SLD-Resolution

**Theorem 4.4**

If there exists a successful SLD-derivation of $P \cup \{Q_0\}$ with $\text{CAS } \theta$, then $P \models Q_0 \theta$.

**Proof.** Let $\xi = Q_0 \Rightarrow \ldots \Rightarrow \square$ be a successful SLD-derivation. Lemma 4.3 (ii) applied to the resultant of level $n$ of $\xi$ implies $P \models Q_0 \theta_1 \ldots \theta_n$ and $Q_0 \theta_1 \ldots \theta_n = Q_0 (\theta_1 \ldots \theta_n \mid_{\text{Var}(Q_0)}) = Q_0 \theta$. 

Foundations of Logic Programming  Declarative Interpretation
Comparison to Intuitive Meaning of Queries

Corollary 4.5

If there exists a successful SLD-derivation of \( P \cup \{Q_0\} \), then \( P \models \exists Q_0 \).

Proof.
Theorem 4.4 implies \( P \models Q_0 \theta \) for some \( \texttt{cas} \ \theta \).

Then, \( P \models Q_0 \theta \)

implies for every interpretation \( l \): if \( l \models P \), then \( l \models Q_0 \theta \)

implies for every interpretation \( l \): if \( l \models P \), then \( l \models \forall(Q_0 \theta) \)

implies for every interpretation \( l \): if \( l \models P \), then \( l \models \exists Q_0 \)

implies \( P \models \exists Q_0 \)
Towards Completeness of SLD-Resolution

To show completeness of SLD-resolution we need to syntactically characterize the set of semantically derivable queries.

The concepts of term models and implication trees serve this purpose.
Term Models

$V$ set of variables, $F$ function symbols, $\Pi$ predicate symbols:

The term algebra $J$ for $F$ is defined as follows:
1. domain $D = TU_{F,V}$
2. mapping $f_j : (TU_{F,V})^n \rightarrow TU_{F,V}$ assigned to every $f \in F^{(n)}$ with
   $$f_j(t_1, ..., t_n) \iff f(t_1, ..., t_n)$$

A term interpretation $I$ for $F$ and $\Pi$ consists of:
1. term algebra for $F$
2. $I \subseteq TB_{\Pi,F,V}$ (set of atoms that are true; equivalent: assignment of a relation $p_I \subseteq (TU_{F,V})^n$
   to every $p \in \Pi^{(n)}$)

$I$ term model of a set $S$ of expressions $:\iff$ $I$ term interpretation and model of $S$
Herbrand Models

The Herbrand algebra \( J \) for \( F \) is defined as follows:
1. domain \( D = HU_F \)
2. mapping \( f_j : (HU_F)^n \rightarrow HU_F \) assigned to every \( f \in F^{(n)} \) with
   \[
   f_j(t_1, \ldots, t_n) \iff f(t_1, \ldots, t_n)
   \]

A Herbrand interpretation \( I \) for \( F \) and \( \Pi \) consists of:
1. Herbrand algebra for \( F \)
2. \( I \subseteq HB_{\Pi,F} \) (set of ground atoms that are true)

\( I \) Herbrand model of a set \( S \) of expressions \( \iff I \) Herbrand interpretation and model of \( S \)

\( I \) least Herbrand model of a set \( S \) of expressions
\( \iff I \) Herbrand model of \( S \) and \( I \subseteq I' \) for all Herbrand models \( I' \) of \( S \)
Implication Trees

implication tree w.r.t. program $P$

$\iff$

- finite tree whose nodes are atoms
- if $A$ is a node with the direct descendants $B_1, \ldots, B_n$ then $A \leftarrow B_1, \ldots, B_n \in \text{inst}(P)$
- if $A$ is a leaf, then $A \leftarrow \in \text{inst}(P)$

$E$ expression, $S$ set of expressions:

- $\text{inst}(E) :\iff$ set of all instances of $E$
- $\text{inst}(S) :\iff$ set of all instances of Elements $E \in S$
- $\text{ground}(E) :\iff$ set of all ground instances of $E$
- $\text{ground}(S) :\iff$ set of all ground instances of Elements $E \in S$
Implication Trees (Example)

Let $P_{\text{add}}$ “add-program”, $n \in \mathbb{N}$, $V$ set of variables, $t \in TU_{\{s,0\},V}$, and

$$\mathcal{T} = \text{add}(t, s^n(0), s^n(t))$$

$$\phantom{\mathcal{T}} \mid$$

$$\text{add}(t, s^{n-1}(0), s^{n-1}(t))$$

$$\vdots$$

$$\vdots$$

$$\text{add}(t, s(0), s(t))$$

$$\phantom{\mathcal{T}} \mid$$

$$\text{add}(t, 0, t)$$

If $t \in HU_{\{s,0\}}$, then $\mathcal{T}$ is ground implication tree w.r.t. $P_{\text{add}}$. 
Implication Trees Constitute Term Model

Lemma 4.7
Consider term interpretation $I$, atom $A$, program $P$

- $I \models A$ iff $\text{inst}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \text{inst}(P)$: if $\{B_1, \ldots, B_n\} \subseteq I$ then $A \in I$

Lemma 4.12
The term interpretation $C(P) :\iff \{A \mid A$ is the root of some implication tree w.r.t. $P\}$ is a model of $P$. 
Ground Implication Trees Constitute Herbrand Model

Lemma 4.26
Consider Herbrand interpretation $I$, atom $A$, program $P$

- $I \models A$ iff $\text{ground}(A) \subseteq I$
- $I \models P$ iff for every $A \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$, $\{B_1, \ldots, B_n\} \subseteq I$ implies $A \in I$

Lemma 4.28
The Herbrand interpretation $\mathcal{M}(P) :\iff \{A \mid A \text{ is the root of some ground implication tree w.r.t. } P\}$ is a model of $P$. 

Foundations of Logic Programming
Declarative Interpretation
Example

Let $P_{add}$ “add-program”, and $V$ set of variables.

The term interpretation
\[
\mathcal{C}(P_{add}) = \{\text{add}(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in TU_{\{s,0\}, V}\}
\]
\[
= \{\text{add}(s^m(v), s^n(0), s^{n+m}(v)) \mid m, n \in \mathbb{N}, v \in V \cup \{0\}\}
\]

and the Herbrand interpretation
\[
\mathcal{M}(P_{add}) = \{\text{add}(t, s^n(0), s^n(t)) \mid n \in \mathbb{N}, t \in HU_{\{s,0\}}\}
\]
\[
= \{\text{add}(s^m(0), s^n(0), s^{n+m}(0)) \mid m, n \in \mathbb{N}\}
\]

are models of $P_{add}$.
Correct Answer Substitutions versus Computed Answer Substitutions (Example)

Let $P_{add}$ “add-program”, and $Q = \text{add}(u, s(0), s(u))$ query.

- $\theta = \{u/s^2(v)\}$ correct answer substitution of $Q$, since $P_{add} \models Q\theta = \text{add}(s^2(v), s(0), s^3(v))$
  (in analogy to slide 13 with $x = s^2(v)$).

- SLD-derivation of $P_{add} \cup \{Q\}$:
  
  \[
  \xrightarrow{\theta_1} \text{add}(u, s(0), s(u)) \xrightarrow{\theta_2} \text{add}(u, 0, u) \xrightarrow{\Box} \text{with } \theta_1 = \{x/u, y/0, z/u\} \text{ and } \theta_2 = \{x/u\},
  \]
  thus $\eta = (\theta_1 \theta_2)|_{\{u\}} = \epsilon$ is a computed answer substitution of $Q$.

- Thus, $Q\eta$ more general than $Q\theta$.

- In fact, no SLD-derivation of $P_{add} \cup \{Q\}$ can deliver correct answer substitution $\theta$. 

Completeness of SLD-Resolution for Implication Trees

Query $Q$ is $n$-deep.

$\iff$

every atom in $Q$ is the root of an implication tree,
and $n$ is the total number of nodes in these trees

Lemma 4.15

Suppose $Q\theta$ is $n$-deep for some $n \geq 0$. Then for every selection rule $\mathcal{R}$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with $\text{CAS} \ \eta$ such that $Q\eta$ is more general than $Q\theta$. 
Completeness of SLD-Resolution (I)

Theorem 4.13
Suppose that $\theta$ is a correct answer substitution of $Q$. Then for every selection rule $\mathcal{R}$ there exists a successful SLD-derivation of $P \cup \{Q\}$ with $\text{cas } \eta$ such that $Q\eta$ is more general than $Q\theta$.

Proof. Let $Q = A_1, \ldots, A_m$. Then: $\theta$ correct answer substitution of $A_1, \ldots, A_m$
implies $P \models A_1\theta, \ldots, A_m\theta$
implies for every interpretation $I$: if $I \models P$, then $I \models A_1\theta, \ldots, A_m\theta$
implies $C(P) \models A_1\theta, \ldots, A_m\theta$ (since $C(P) \models P$ by Lemma 4.12)
implies $\text{inst}(A_i\theta) \subseteq C(P)$ for every $i = 1, \ldots, m$ (by Lemma 4.7)
implies $A_i\theta \in C(P)$ for every $i = 1, \ldots, m$
implies $A_1\theta, \ldots, A_m\theta$ is $n$-deep for some $n \geq 0$ (by def. of $C(P)$)
implies claim (by Lemma 4.15)
Completeness of SLD-Resolution (II)

Corollary 4.16
Suppose $P \vdash \exists Q$.
Then there exists a successful SLD-derivation of $P \cup \{Q\}$.

Proof. $P \vdash \exists Q$
implies $P \vdash Q\theta$ for some substitution $\theta$
implies $\theta$ correct answer substitution of $Q$
implies claim (by Theorem 4.13)
Least Herbrand Model

Theorem 4.29 \( \mathcal{M}(P) \) is the least Herbrand model of \( P \).

Proof. Let \( I \) be a Herbrand model of \( P \) and let \( A \in \mathcal{M}(P) \).

We prove \( A \in I \) by induction on the number \( i \) of nodes in the ground implication tree w.r.t. \( P \) with root \( A \). Then \( \mathcal{M}(P) \subseteq I \).

\( i = 1: \) A leaf implies \( A \leftarrow \in \text{ground}(P) \)
\implies I \models A \text{ (since } I \models P) \)
\implies A \in I \)

\( i \rightarrow i+1: \) A has direct descendants \( B_1, \ldots, B_n \) (roots of subtrees)
implies \( A \leftarrow B_1, \ldots, B_n \in \text{ground}(P) \) and \( B_1, \ldots, B_n \in I \) (induction hypothesis)
implies \( A \leftarrow B_1, \ldots, B_n \in \text{ground}(P) \) and \( I \models B_1, \ldots, B_n \)
implies \( I \models A \text{ (since } I \models P) \)
implies \( A \in I \)
Ground Equivalence

Theorem 4.30  For every ground atom $A$: $P \models A$ iff $\mathcal{M}(P) \models A$.

Proof. “⇒”: $P \models A$ and $\mathcal{M}(P) \models P$ implies $\mathcal{M}(P) \models A$ (semantic consequence).

“⇐”: Show for every interpretation $I$: $I \models P$ implies $I \models A$.

Let $I_H = \{A \mid A$ ground atom and $I \models A\}$ Herbrand interpretation.

$I \models P$

implies $I \models A \leftarrow B_1, \ldots, B_n$ for all $A \leftarrow B_1, \ldots, B_n \in \text{ground}(P)$

implies if $I \models B_1, \ldots, I \models B_n$ then $I \models A$ for all ...

implies if $B_1 \in I_H, \ldots, B_n \in I_H$ then $A \in I_H$ for all ... (Def. $I_H$)

implies $I_H \models P$ (by Lemma 4.26; thus $I_H$ Herbrand model)

implies $A \in I_H$ (since $A \in \mathcal{M}(P)$ and $\mathcal{M}(P)$ least Herbrand model)

implies $I \models A$ (by Def. $I_H$)
Complete Partial Orderings

Let \((\mathcal{A}, \sqsubseteq)\) be a partial ordering (cf. Slide 18 for Chapter 2).

- **a least element** of \(X \subseteq \mathcal{A}\)
  \[\iff a \in X, a \sqsubseteq x \text{ for all } x \in X\]

- **a least upper bound** of \(X \subseteq \mathcal{A}\) (Notation: \(a = \sqcup X\))
  \[\iff a \in \mathcal{A}, x \sqsubseteq a \text{ for all } x \in X \text{ and } a \text{ is the least element of } \mathcal{A} \text{ with this property}\]

\((\mathcal{A}, \sqsubseteq)\) complete partial ordering (\textit{cpo}) \(\iff\)

- \(\mathcal{A}\) contains a least element (denoted by \(\emptyset\))

- for every increasing sequence \(a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots\) of elements of \(\mathcal{A}\),
  the set \(X = \{a_0, a_1, a_2, \ldots\}\) has a least upper bound
Some Properties of Operators

Let \((\mathcal{A}, \sqsubseteq)\) be a \textit{cPO}.

operator \(T: \mathcal{A} \rightarrow \mathcal{A}\) monotonic

\(\iff I \sqsubseteq J\) implies \(T(I) \sqsubseteq T(J)\)

operator \(T: \mathcal{A} \rightarrow \mathcal{A}\) finitary

\(\iff\) for every infinite sequence \(I_0 \sqsubseteq I_1 \sqsubseteq \ldots\),

\[\bigcup_{n=0}^{\infty} T(I_n) \text{ exists \ and \ } T\left(\bigcup_{n=0}^{\infty} I_n\right) \sqsubseteq \bigcup_{n=0}^{\infty} T(I_n)\]

operator \(T: \mathcal{A} \rightarrow \mathcal{A}\) continuous \(\iff T\) monotonic and finitary

\(I\) pre-fixpoint of \(T\) : \(\iff T(I) \sqsubseteq I\)

\(I\) fixpoint of \(T\) : \(\iff T(I) = I\)
Iterating Operators

Let \((\mathcal{A}, \sqsubseteq)\) be a cpo, \(T: \mathcal{A} \to \mathcal{A}\), and \(l \in \mathcal{A}\).

- \(T^\uparrow 0 (l) :\iff l\)
- \(T^\uparrow (n + 1) (l) :\iff T(T^\uparrow n (l))\)
- \(T^\uparrow w (l) :\iff \bigsqcup_{n=\omega}^\rho. T \uparrow n (l)\)

\(T^\uparrow a :\iff T^\uparrow a (\emptyset) \) (for \(a = 0, 1, 2, \ldots, w\))

By the definition of a cpo:
If the sequence \(T^\uparrow 0 (l), T^\uparrow 1 (l), T^\uparrow 2 (l), \ldots\) is increasing, then \(T^\uparrow w (l)\) exists.

**Theorem 4.22**

If \(T\) is a continuous operator on a cpo, then \(T^\uparrow w\) exists and is the least prefixpoint of \(T\) and the least fixpoint of \(T\).
Consequence Operator

Consider the \( \text{cfo} \) (\( \{l \mid l \text{ Herbrand interpretation}\}, \subseteq \)).

Let \( P \) be a program and \( l \) a Herbrand interpretation. Then
\[
T_P(l) :\Leftrightarrow \{A \mid A \leftarrow B_1, \ldots, B_n \in \text{ground}(P), \{B_1, \ldots, B_n\} \subseteq l\}
\]

Lemma 4.33

(i) \( T_P \) is finitary.

(ii) \( T_P \) is monotonic.
$T_P$-Characterization

Lemma 4.32

A Herbrand interpretation $I$ is a model of $P$ iff

$$T_P(I) \subseteq I$$

Proof.

$I \models P$

iff for every $A \leftarrow B_1, ..., B_n \in \text{ground}(P)$:

$$\{B_1, ..., B_n\} \subseteq I \text{ implies } A \in I \quad \text{(by Lemma 4.26)}$$

iff for every ground atom $A$: $A \in T_P(I)$ implies $A \in I$

iff $T_P(I) \subseteq I$
Characterization Theorem

Theorem 4.34

\[ \mathcal{M}(P) \]

(i)

= least Herbrand model of \( P \)  
(ii)

= least pre-fixpoint of \( T_P \)  
(iii)

= least fixpoint of \( T_P \)  
(iv)

= \( T_P \uparrow^w \)  
(v)

= \( \{A \mid A \text{ ground atom, } P \models A\} \)  
(vi)
Success Sets

success set of a program $P$ : $\iff$

$\{A \mid A \text{ ground atom, } \exists \text{ successful SLD-derivation of } P \cup \{A\} \}$

Theorem 4.37

For a ground atom $A$, the following are equivalent:

(i) $\mathcal{M}(P) \models A$

(ii) $P \models A$

(iii) Every SLD-tree for $P \cup \{A\}$ is successful

(iv) $A$ is in the success set of $P$
Objectives

- Algebras (which provide a semantics of terms)
- Interpretations (which provide a semantics of programs)
- Soundness of SLD-resolution
- Completeness of SLD-resolution
- Least Herbrand models
- Computing least Herbrand models