A Semantic Approach to a Logical Framework

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Contents

- Background / Introduction to logical frameworks
- A semantic framework for studying a logical framework
- A class of representable logics

Judgements

Immanuel Kant (1800)

Introduces the notion of judgement.

A **judgement** is an object of knowledge, *e.g.*,

- It is raining;
- The sky is blue;
- ϕ has a proof;
- ψ is a proposition.

Higher-Order Judgements

Per Martin-Löf (1983)
Develops Kant's notion of judgement to include two higher-order judgements:

- The **Hypothetical**

$$\mathsf{J}_1, \ldots, \mathsf{J}_n \vdash \mathsf{K};$$

- The **General**

$$\Lambda x \in C \,.\, \mathsf{J}(x).$$

These combine to give the hypothetico-general judgement:

$$\Lambda x \in C \,.\, \mathsf{J}_1(x), \ldots, \mathsf{J}_n(x) \vdash \mathsf{K}(x).$$

Natural Deduction Rules as Hypothetico-general Judgements

(Martin-Löf)

Hypothetico-general judgements are sufficient to describe any natural deduction inference rule.

Example:

$$\dfrac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \supset \psi} \supset I$$

Hypothetico-general Judgement:
$\Lambda \phi, \psi \in Prop\,.\,(\mathsf{Proof}(\phi) \vdash \mathsf{Proof}(\psi)) \vdash \mathsf{Proof}(\phi \supset \psi)$

Logical Framework

A **logical framework** is a formal meta-theory in which we can represent logical systems viewed in terms of judgements.

$$Framework = Language + Representation$$

We use a typed λ-calculus to represent a logical system whose inference rules are hypothetico-general judgements.

Language

The choice of language is based on the following:

- It needs to contain types which correspond to hypothetico-general judgements;
- The dependent type $\Pi x : A \, . \, B$ corresponds to both hypothetical and general judgements;
- We keep the language minimal since we wish to study aspects of the representation.

$\lambda\Pi$ is a dependently typed λ-calculus with a signature in propositions-as-types correspondence with the $\{\forall, \supset\}$-fragment of minimal first-order logic.

We take $\lambda\Pi$ to be our language.

Representation Mechanism

We take our representation mechanism to be that of **judgements-as-types**:

- Judgements are represented as types;
- Proofs are represented as terms whose type is the representation of the judgement they prove;
- The judgement Proof(ϕ) corresponds to the type Proof$(\phi) : \mathrm{Type}$;
- The hypothetical judgement J $\vdash$ K corresponds to J $\rightarrow$ K;
- The general judgement $\Lambda x \in C \,.\, \mathsf{J}(x)$ corresponds to $\Pi x \in C \,.\, \mathsf{J}(x)$.

Example

$$\frac{\mathsf{Proof}(\phi \supset \psi) \quad \mathsf{Proof}(\phi)}{\mathsf{Proof}(\psi)} \supset E$$

Encoded as:

$$\supset E : \Pi \phi, \psi : Prop\,.\, \mathsf{Proof}(\phi) \rightarrow \mathsf{Proof}(\phi \supset \psi) \rightarrow \mathsf{Proof}(\psi)$$

Instantiate:

$$(\supset E)(p)(q) : \mathsf{Proof}(p) \rightarrow \mathsf{Proof}(p \supset q) \rightarrow \mathsf{Proof}(q)$$

Given $\Phi : \mathsf{Proof}(p)$ and $\Psi : \mathsf{Proof}(p \supset q)$, we have:

$$(\supset E)(p)(q)(\Phi)(\Psi) : \mathsf{Proof}(q)$$

Encoding

Given a proof

$$X, \underbrace{y_1 : \mathsf{j}_1(\phi), \ldots, y_n : \mathsf{j}_n(\phi_n)}_{\Delta} \vdash \delta : \mathsf{j}(\phi)$$

in the object-logic, this is represented as a derivation

$$\Gamma_X, \Gamma_\Delta \vdash_\Sigma M_\delta : \mathsf{j}(\phi)$$

in the encoded logic.

Adequate Representation

We are interested in the relationship between consequence in the object-logic and consequence in the encoded-logic.
An encoding is **full** if for every proof in the object-logic there is a corresponding derivation in the encoded logic, *i.e.*,

$$X, \Delta \vdash \delta : \mathsf{j}(\phi) \Rightarrow \Gamma_X, \Gamma_\Delta \vdash_\Sigma M_\delta : \mathsf{j}(\phi).$$

An encoding is **faithful** if for every proof in the encoded-logic there is a corresponding derivation in the object-logic *i.e.*,

$$\Gamma_X, \Gamma_\Delta \vdash_\Sigma M_\delta : \mathsf{j}(\phi) \Rightarrow X, \Delta \vdash \delta : \mathsf{j}(\phi).$$

An encoding is **adequate** if it is both full and faithful.

Proving Adequacy

- Proving that an encoding is full is easy proof-theoretically.
- It is an induction on the structure of proofs in the object-logic.
- Proving that an encoding is faithful is hard proof-theoretically.
- We have to show that any derivation in the encoded-logic is the representation of a proof in the object-logic.
- Proving faithfulness should be easy semantically.
- It should follow from the interpretation of the consequence relation of the encoded-logic.

Road Map

- We will provide a semantic framework which will allow us to prove faithfulness in a natural and uniform way.
- We will use this framework to prove that classes of logics are representable in a logical framework.

A Semantic Analysis

Providing the object-logic and encoded-logic with a semantics gives us the following picture:

$$\begin{tikzcd}
\text{Model of Object-logic} \arrow[d, leftrightarrow] & \text{Model of Encoded-logic} \arrow[l] \arrow[d, leftrightarrow] \\
\text{Object-logic} \arrow[r] & \text{Encoded-logic}
\end{tikzcd}$$

The faithfulness proof proceeds as follows:

(i) $\Gamma_X, \Gamma_\Delta \vdash_\Sigma M_\delta : \mathsf{j}(\phi)$ is interpreted in the model of the encoded-logic by soundness.

(ii) We use the morphism induced by the representation mechanism to find the corresponding interpretation in the model of the object-logic.

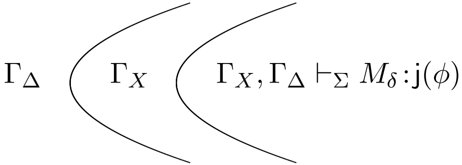
(iii) Completeness gives us the proof $X, \Delta \vdash \delta : \mathsf{j}(\phi)$ in the object-logic.

Kripke Models

- We provide a Kripke semantics since $\lambda\Pi$ can be viewed as being intuitionistic.
- We have a slightly unusual notion of Kripke partiality.
- We usually think of a proof as being defined at all worlds in a Kripke model.
- In our models a proof is not necessarily defined at a world.
- Worlds are collections of proof variables.
- A proof can only be interpreted at a world which contains all the proof variables corresponding to its assumptions.

Kripke Model for encoded-logic

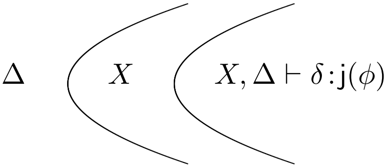
- The **Kripke model for the encoded-logic** is based on Pitts' presentation of Cartmell's categories with attributes.
- We interpret not just proofs but also consequence.



Γ_Δ $\quad$ Γ_X $\quad$ $\Gamma_X, \Gamma_\Delta \vdash_\Sigma M_\delta : \mathsf{j}(\phi)$

Kripke Model for the object-logic

- The **Kripke model for the object-logic** is based on the hyperdoctrine model of Lawvere and Seely.
- The model is similar to the Kripke model for the encoded-logic.



Δ X $X, \Delta \vdash \delta : \mathsf{j}(\phi)$

Judgements-as-Types Morphisms

- We construct a category of Kripke models. Objects are Kripke models of the object-logic and the encoded-logic.
- We restrict our analysis to surjective encodings, since we want every element of the signature to have come from the object-logic.

Theorem

If we have a surjective, full encoding together with sound and complete Kripke models for the object-logic and the encoded-logic. Then there is an epimorphism between these models.

- This epimorphism preserves the interpretation.

Propositions-as-Types Isomorphism

- We obtain an isomorphism between Kripke models as a special case.
- Change the representation mechanism to propositions-as-types.
- Take the $\{\forall, \supset\}$-fragment of minimal first-order logic as the object-logic.
- We call this isomorphism of Kripke models the **propositions-as-types isomorphism**.

Summary

- We have sketched the semantic framework in which we can prove faithfulness.
- Faithfulness proofs have been reduced to proofs of soundness and completeness and the surjectivity of the encoding.
- Given this framework we wish to use it to develop classes of representable logics. This will be the remainder of the presentation.

Labelled Deductive Systems

- We sketch the **labelled deductive systems** of Basin, Matthews and Viganò.
- Each formula is labelled and relations between labels are allowed to be included in inference rules.
- A class of labelled deductive systems consists of a **base system** and a **labelling algebra**.
- A base system is a collection of inference rules for all the connectives which may or may not involve relations.
- A labelling algebra consists of inference rules for relations.

A Class of Representable Modal Logics

- We have the base system K, which corresponds to the modal logic K.
- K contains the usual natural deduction rules for $\perp$ and $\supset$ with the judgement having two arguments $\mathsf{Proof}(x, \phi)$.
- We also have rules for $\Box$ introduction and elimination:

$$\dfrac{\begin{array}{c}[\mathsf{Related}(x, y)] \\ \vdots \\ \mathsf{Proof}(y, \phi)\end{array}}{\mathsf{Proof}(x, \Box\phi)}\ \Box I \qquad \dfrac{\mathsf{Proof}(x, \Box\phi) \quad \mathsf{Related}(x, y)}{\mathsf{Proof}(y, \phi)}\ \Box E$$

- The judgement Related means we do not have to have a side condition.

A Class of Representable Modal Logics ctd

- The labelling algebra is based on the relational counterpart to the **generalized Geach axiom schema**:

  $$(G) : \Diamond^i \Box^m \phi \supset \Box^j \Diamond^n \phi$$

  which describes a large class of modal logics.

- This corresponds to the (i, j, m, n)-**convergency axioms**:

  $$\forall x \forall y \forall z (xR^i y \wedge xR^j z \supset \exists u (yR^m u \wedge zR^n u))$$

  where $xR^0 y$ means $x = y$ and $xR^{i+1} y$ means $\exists v (xRv \wedge vR^i y))$.

- We restrict to the case where $m = n = 0$ implies $i = j = 0$.

Horn Relational Theories

- A theory corresponding to a collection of restricted (i, j, m, n)-convergency axioms has a corresponding Horn relation theory which conservatively extends it.
- A Horn relational theory is generated by Horn relational rules:

$$\frac{\mathsf{Related}(s_1, t_1) \quad \cdots \quad \mathsf{Related}(s_m, t_m)}{\mathsf{Related}(s, t)}$$

- The Horn rules

$$\frac{}{\mathsf{Related}(x, x)}\, refl \quad \frac{\mathsf{Related}(x, y) \quad \mathsf{Related}(x, z)}{\mathsf{Related}(z, y)}\, eucl \text{ give}$$

$S5$.

- These correspond to adding $T : \Box\phi \supset \phi$ and $5 : \Diamond\phi \supset \Box\Diamond\phi$ to K.

Result

Theorem

The class of propositional modal logics just described can be adequately encoded in a logical framework.

Ongoing Work

- Add equality and so deal with the wider class of convergency axioms.
- Basin, Matthews and Viganò describe other classes of logics, these can be represented in a logical framework but they contain substructural logics which cause problems.
- Hilbert-type systems can also fit into the framework we have described above. Which logics can be characterized by combinations of Hilbert and natural deduction rules?
- Can this analysis be carried out for substructural logics? We would need to adapt our analysis to the relevant and linear logical frameworks.

Acknowledgments

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