Elliptic Functions

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1 Introduction

I have used the following notation in this essay: the set of all complex numbers is denoted by $\mathbb{C}$ and the set of all real numbers is denoted by $\mathbb{R}$. Also where theorems have been taken from books, the proofs have been expanded and added to for the purpose of showing understanding.

In this essay I will start by analysing a problem in classical mechanics which has a solution comprising of elliptic functions. I will then do some analysis on the elliptic functions which arise from the problem and will discuss briefly how this analysis can be applied to the problem and suggest how it can be used to predict other solutions. I will then look at the general theory of elliptic functions and then study another specific class before finally looking briefly at the algebraic theory of elliptic functions.

2 Analysis of motion of a Pendulum

In this section I am going to analyse the motion of a simple pendulum with no approximations. This should give motivation to the theory and also allow further analysis of the solution to be done after the theory.

The following analysis is taken from [4] P.333-334.

Consider the following pendulum with mass m, length a and angle from its rest position θ.

In this case the total kinetic energy + total potential energy = the total energy.

Hence

$$\frac{1}{2}ma^2\left(\frac{d\theta}{dt}\right)^2 - mga\cos\theta = E \tag{1}$$

Now allow the motion to be oscillatory with amplitude α. So if $\theta = \pm\alpha$, $\frac{d\theta}{dt} = 0$ which implies that $E = -mga\cos\alpha$. Putting this into (1) gives

$$\begin{aligned}
\tfrac{1}{2}ma^2\big(\tfrac{d\theta}{dt}\big)^2 - mga\cos\theta &= -mga\cos\alpha \\
\Rightarrow \tfrac{1}{2}a^2\big(\tfrac{d\theta}{dt}\big)^2 &= g(\cos\theta - \cos\alpha) \\
\Rightarrow \big(\tfrac{d\theta}{dt}\big)^2 &= g(\cos\theta - \cos\alpha) \\
\Rightarrow \big(\tfrac{d\theta}{dt}\big)^2 &= 2p^2(\cos\theta - \cos\alpha) \text{ where } p^2 = \tfrac{g}{a} \\
\Rightarrow \big(\tfrac{d\theta}{dt}\big)^2 &= 2p^2(1 - 2\sin^2\tfrac{\theta}{2} - 1 + 2\sin^2\tfrac{\alpha}{2}) \text{ by trigonometric identities} \\
\Rightarrow \big(\tfrac{d\theta}{dt}\big)^2 &= 4p^2(\sin^2\tfrac{\alpha}{2} - \sin^2\tfrac{\theta}{2}) \quad (2)
\end{aligned}$$

Now define ϕ by $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi$ (3)
Differentiating (3) with respect to time yields $\frac{1}{2}\cos\frac{\theta}{2}(\frac{d\theta}{dt}) = \sin\frac{\alpha}{2}\cos\phi(\frac{d\phi}{dt})$ (4)
Now multiply (2) by $\cos\frac{\theta}{2}$ to get

$$\cos^2\frac{\theta}{2}\left(\frac{d\theta}{dt}\right)^2 = 4p^2\cos^2\frac{\theta}{2}\big(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\big)$$

Substituting from (3) and (4) gives

$$\begin{aligned}
4\sin^2\tfrac{\alpha}{2}\cos^2\phi\big(\tfrac{d\phi}{dt}\big)^2 &= 4p^2\cos^2\tfrac{\theta}{2}(\sin^2\tfrac{\alpha}{2} - \sin^2\tfrac{\alpha}{2}\sin^2\phi) \\
\Rightarrow \sin^2\tfrac{\alpha}{2}\cos^2\phi\big(\tfrac{d\phi}{dt}\big)^2 &= p^2\cos^2\tfrac{\theta}{2}(\sin^2\tfrac{\alpha}{2} - \sin^2\tfrac{\alpha}{2} + \cos^2\phi\sin^2\tfrac{\alpha}{2}) \\
&\quad \text{by trigonometric identities} \\
\Rightarrow \sin^2\tfrac{\alpha}{2}\cos^2\phi\big(\tfrac{d\phi}{dt}\big)^2 &= p^2\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\alpha}{2}\cos^2\phi \\
\Rightarrow \cos^2\phi\big(\tfrac{d\phi}{dt}\big)^2 &= p^2\cos^2\tfrac{\theta}{2}\cos^2\phi \text{ since } \sin^2\tfrac{\alpha}{2} \neq 0 \\
\Rightarrow \cos^2\phi\big(\tfrac{d\phi}{dt}\big)^2 &= p^2(1 - \sin^2\tfrac{\phi}{2})(1 - \sin^2\tfrac{\phi}{2}) \text{ by trigonometric identities} \\
\Rightarrow \quad (5) \quad \cos^2\phi\big(\tfrac{d\phi}{dt}\big)^2 &= p^2(1 - \sin^2\phi)(1 - \sin^2\tfrac{\alpha}{2}\sin^2\phi) \text{ by } \quad (3)
\end{aligned}$$

Now let $y = \sin\phi \Rightarrow \frac{dy}{dt} = \cos\phi(\frac{d\phi}{dt})$ and $k = \sin\frac{\alpha}{2}$
So (5) becomes $(\frac{dy}{dt})^2 = p^2(1 - y^2)(1 - k^2y^2)$
Now let $x = pt$
So $\frac{dx}{dt} = p$
Then $\frac{dy}{dx} = (\frac{dy}{dt})(\frac{dt}{dx}) = (\frac{dy}{dt})(\frac{1}{p}) = \sqrt{(1 - y^2)(1 - k^2y^2)}$
So $y = \int \sqrt{(1 - y^2)(1 - k^2y^2)}dy$
Since $\sqrt{(1 - y^2)(1 - k^2y^2)}$ is integrable. We may write the solution as $y = sn(x + c)$.

hence the general motion of a simple pendulum with no approximations is $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2} sn[p(t - t_0)]$ where t_0 is a constant.

sn is normally defined for $0 < k < 1$. If we let $k \to 0$, we get the solution as a function of sin. This corresponds to the first approximation of the simple pendulum. If we let $k \to 1$, then we get a solution which is a function of tan. This corresponds to the pendulum doing a full revolution, i.e it rotates towards the unstable equilibrium point at the top but never reaches it in finite time.

These are both extreme cases of the simple pendulum. In reality it does something in between and so it is sensible to assume that $0 < k < 1$ gives a realistic solution.

3 Jacobian Elliptic Functions

In this section, I will look at the sn function and the other functions closely related to it. These functions are called the Jacobian Elliptic Functions.

I will now look at the Jacobian Elliptic Functions.

I will start by defining sn and the other Jacobian Elliptic Functions.

DEFINITIONS

*D*efine sn as the function which satisfies the following equation:

$$u = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad 0 < k < 1 \quad (6)$$

and $sn() = 0$.

Define cn as the function which satisfies the following equation:

$$cn^2(u) = 1 - sn^2(u) \quad (7)$$

and $cn(0) = 1$.

Define dn as the function which satisfies the following equation:

$$dn^2(u) = 1 - k^2 sn^2(u) \quad (8)$$

and $dn(0) = 1$. ■

If $k=0$ in the definition of sn, (6) reduces to $u=\int_0^y \frac{dt}{\sqrt{(1-t^2)}}$ and the solution is $y=\sin u$. So sn is just a generalisation of sin. This leads to cn becoming cos when $k=0$ and dn becoming 1.

If in the definition of sn, t is replaced by $-t$, we get the sign of y changed. So sn is an odd function. From our definition of cn it follows that $cn(-u)=\pm cn(u)$. Put $u=0$ to see that $cn(-u)=cn(u)$ and hence even.

I will now consider some of the relationships between sn, cn and dn[1].

From (6) we get $(\frac{d}{du}sn(u))^2=(1-sn^2(u))(1-k^2sn^2(u))$ (9)

$$\begin{aligned}
\text{so } \left(\tfrac{d}{du}sn(u)\right)^2 &= cn^2(u)dn^2(u)\\
\Rightarrow \tfrac{d}{du}sn(u) &= cn(u)dn(u)\\
\text{Differentiating (7) gives } 2cn(u)\tfrac{d}{du}cn(u) &= -2sn(u)\tfrac{d}{du}sn(u)\\
\text{so } cn(u)\tfrac{d}{du}cn(u) &= -sn(u)cn(u)dn(u)\\
\Rightarrow \tfrac{d}{du}cn(u) &= -sn(u)dn(u)
\end{aligned}$$

Similarly (8) gives $\frac{d}{du}dn(u)=-k^2sn(u)cn(u)$

From the earlier analogy with sin and cos it would seem sensible to assume that there exist addition formulae. Here is the necessary theorem:

THEOREM 1

*W*e have

$$\begin{aligned}
sn(u+v) &= \tfrac{sn(u)cn(v)dn(v)+sn(v)cn(u)dn(u)}{1-k^2sn^2(u)sn^2(v)}\\
cn(u+v) &= \tfrac{cn(u)cn(v)-sn(u)sn(v)dn(u)dn(v)}{1-k^2sn^2(u)sn^2(v)}\\
dn(u+v) &= \tfrac{dn(u)dn(v)-k^2sn(u)sn(v)dn(u)cn(v)}{1-k^2sn^2(u)sn^2(v)}
\end{aligned}$$

Proof.

There are many alternate proofs to this theorem but this one comes from [6] P.495-497.

Let $u+v=\alpha$ where u and v vary but α is constant such that $\frac{dv}{du}=-1$.

From (9) we get $\left(\frac{dsn(u)}{du}\right)^2=(1-sn^2(u))(1-k^2sn^2(u))$ (10)

Differentiating $sn(v)$ and squaring gives $\left(\frac{dsn(v)}{du}\right)^2\left(\frac{dv}{du}\right)^2=(1-sn^2(v))(1-k^2sn^2(v))$ (11)

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[1] If you put $k=0$ in the following relationships you will obtain the standard trigonometric formulae

Now differentiate (10) by u to get

$$\begin{aligned} 2\tfrac{dsn(u)}{du}.\tfrac{d^2sn(u)}{du^2} &= -2sn(u)k^2\tfrac{dsn(u)}{du} - sn(u)\tfrac{dsn(u)}{du} + 4k^2sn^3(u)\tfrac{dsn(u)}{du} \\ \Rightarrow \tfrac{d^2sn(u)}{du^2} &= -k^2sn(u) - sn(u) + 2k^2sn^3(u) \\ &= -(1+k^2)sn(u) + 2k^2sn^3(u) \quad (12) \quad \text{for } cn(u) \neq 0 \text{ and } dn \neq 0 \end{aligned}$$

Differentiate (11) by u to get

$$\begin{aligned} 2\big(\tfrac{dsn(v)}{dv}\big)\big(\tfrac{d^2sn(v)}{du^2}\big)\big(\tfrac{dv}{du}\big)^2 &= -2sn(v)\tfrac{dsn(v)}{du}\big(\tfrac{dv}{du}\big)^2 - 2k^2sn(v)\tfrac{dsn(v)}{du}\big(\tfrac{dv}{du}\big)^2 + 4k^2sn^3(v)\tfrac{dsn(v)}{du}\big(\tfrac{dv}{du}\big)^2 \\ \Rightarrow \tfrac{d^2sn(v)}{du^2} &= -sn(v) - k^2sn(v) + 2k^2sn^3(v) \\ &= -(1+k^2)sn(v) + 2k^2sn^3(v) \quad (13) \quad \text{for } cn(u) \neq 0 \text{ and } dn \neq 0 \end{aligned}$$

$$\begin{array}{crcl} (10) \times sn^2(v) & \big(\tfrac{dsn(u)}{du}\big)^2 sn^2(v) &=& sn^2(v) - (1+k^2)sn^2(u)sn^2(v) + k^2sn^4(u)sn^2(u) \\ (11) \times sn^2(u) & \big(\tfrac{dsn(v)}{du}\big)^2 sn^2(u) &=& sn^2(u) - (1+k^2)sn^2(v)sn^2(u) + k^2sn^4(v)sn^2(u) \\ \text{subtract} & \big(\tfrac{dsn(u)}{du}\big)^2 sn^2(v) - \big(\tfrac{dsn(v)}{du}\big)^2 sn^2(u) &=& sn^2(v) - sn^2(u) + k^2sn^4(u)sn^2(v) - k^2sn^4(v)sn^2(u) \\ & &=& (sn^2(v) - sn^2(u))(1 - k^2sn^2(u)sn^2(v)) \quad (14) \\ (12) \times sn(v) & \tfrac{d^2sn(u)}{du^2}sn(v) &=& -(1+k^2)sn(u)sn(v) + 2k^2sn^3(u)sn(v) \\ (13) \times sn(u) & \tfrac{d^2sn(v)}{du^2}sn(u) &=& -(1+k^2)sn(v)sn(u) + 2k^2sn^3(v)sn(u) \\ \text{subtract} & \tfrac{d^2sn(u)}{du^2}sn(v) - \tfrac{d^2sn(v)}{du^2} &=& 2k^2sn^3(u)sn(v) - s2k^2sn^3(v)sn(u) \\ & &=& 2k^2sn(u)sn(v)(sn^2(u) - sn^2(v)) \quad (15) \\ (15) \div (14) & \dfrac{\frac{d^2sn(v)}{du^2}sn(v) - \frac{d^2sn(v)}{du^2}sn(u)}{\left(\frac{dsn(u)}{du}\right)^2 sn(v) - \left(\frac{dsn(v)}{du}\right)^2} &=& \dfrac{2k^2sn(u)sn(v)(sn^2(u)-sn^2(v))}{(sn^2(v)-sn^2(u))(1-k^2sn^2(u)sn^2(v))} \\ \Rightarrow & \dfrac{\frac{d}{du}\left(\frac{dsn(u)}{du}dn(v) - \frac{dsn(v)}{du}sn(u)\right)}{\left(\frac{dsn(u)}{du}sn(v) - \frac{dsn(v)}{du}\right)} &=& \dfrac{\frac{d}{du}\left(1-k^2sn^2(u)sn^2(v)\right)}{(1-k^2sn^2(u)sn^2(v))} \end{array}$$

Integrating yields

$$\frac{\frac{dsn(u)}{du}sn(v) - \frac{dsn(v)}{du}sn(u)}{1 - k^2sn^2(u)sn^2(v)} = C \text{ , a constant}$$

but we already know $\frac{dsn(u)}{du}$ and $\frac{dsn(v)}{du}$, so

$$\frac{cn(u)dn(u)sn(v) + cn(v)dn(v)sn(u)}{1 - k^2sn^2(u)sn^2(v)} = C$$

This can be written as a function of two variables $f(u+v)$

Let $v = 0$

$$\frac{cn(u)dn(u)sn(0) + cn(0)dn(0)sn(u)}{1 - k^2sn^2(u)sn^2(0)} = f(u)$$

So f is sn.
Hence $sn(u+v) = \frac{cn(u)dn(u)sn(v)+cn(v)dn(v)sn(u)}{1-k^2sn^2(u)sn^2(v)}$
By definition $cn^2(u) = 1 - sn^2(u)$, so

$$cn^2(u+v) = 1 - sn^2(u+v)$$

$$\begin{aligned}
\Rightarrow\ & (1-k^2sn^2(u)sn^2(v))sn^2(u+v) = (1-k^2sn^{(}u)sn^2(v))^2(1-sn^2(u+v)) \\
=\ & (1-k^2sn^2(u)sn^2(v))^2\left(1-\left(\tfrac{sn(u)cn(v)dn(v)+sn(v)cn(u)dn(u)}{1-k^2sn^2(u)sn^2(v)}\right)^2\right) \\
=\ & (1-k^2sn^2(u)sn^2(v))^2 - (sn(u)cn(v)dn(v)+sn(v)cn(u)dn(u))^2 \\
=\ & 1-2k^2sn^2(u)sn^2(v)+k^4sn^4(u)sn^4(v)-sn^2(u)cn^2(v)dn^2(v) \\
& -2sn(u)cn(v)dn(v)sn(u)cn(u)dn(u)-sn^2(v)cn^2(u)dn^2(u) \\
=\ & 1-2k^2sn^2(u)sn^2(v)+k^4sn^4(u)sn^4(v)-sn^2(u)(1-sn^2(v))(1-k^2sn^2(u)) \\
& -sn^2(v)(1-sn^2(u))(1-sn^2(v))-2sn(u)cn(v)dn(v)sn(v)cn(u)dn(u) \\
=\ & 1-2k^2sn^2(u)sn^2(v)+k^4sn^4(u)sn^4(v)-sn^2(u)+k^2sn^2(u)sn^2(v) \\
& +sn^2(v)sn^2(u)-k^2sn^4(v)sn^4(u)-sn^2(v)+k^2sn^2(u)sn^2(v) \\
& +sn^2(v)sn^2(u)-k^2sn^4(u)sn^2(v)-2sn(u)sn(v)cn(u)cn(v)dn(u)dn(v) \\
=\ & 1+k^4sn^4(u)sn^4(v)-k^2sn^4(u)sn^2(v)-k^2sn^4(v)sn^2(u)-sn^2(u)-sn^2(v) \\
& +2sn^2(v)sn^2(u)-2sn(u)cn(u)dn(u)sn(v)cn(v)dn(v) \\
=\ & sn^2(u)sn^2(v)(1-k^2sn^2(u))(1-k^2sn^2(u))+(1-sn^2(u))(1-sn^2(v)) \\
& -2sn(u)cn(u)dn(u)sn(v)cn(v)dn(v) \\
=\ & cn^2(u)cn^2(v)+sn^2(u)sn^2(v)dn^2(u)dn^2(v)-2sn(u)sn(v)cn(u)cn(v)dn(u)dn(v) \\
=\ & (cn(u)cn(v)-sn(u)sn(v)dn(u)dn(v))^2 \\
\Rightarrow\ & cn(u+v) = \pm\tfrac{cn(u)cn(v)-sn(u)sn(v)dn(u)dn(v)}{1-k^2sn^2(u)sn^2(v)}
\end{aligned}$$

Let $u = 0$

$$\begin{aligned}
cn(v) &= \pm\tfrac{cn(0)cn(v)-sn(0)sn(v)dn(0)dn(v)}{1-k^2sn^2(0)sn^2(v)} \\
&= \pm cn(v)
\end{aligned}$$

So have to take the positive term.
Hence $cn(u+v) = \frac{cn(u)cn(v)-sn(u)sn(v)dn(u)dn(v)}{1-k^2sn^2(u)sn^2(v)}$
The same method is used to obtain the remaining formula.

From the definition, $dn^2(u+v) = 1 - k^2 sn^2(u+v)$

$$\begin{array}{rl}
\Rightarrow & (1-k^2sn^2(u)sn^2(v))^2dn^2(u+v) = (1-k^2sn^2(u)sn^2(v))^2(1-k^2sn^2(u+v)) \\
= & (1-k^2sn^2(u)sn^2(v))^2 - k^2(sn(u)cn(v)dn(v)+sn(v)cn(u)dn(u))^2 \\
= & 1-2k^2sn(u)cn(v)dn(v)sn(v)cn(u)dn(u) - k^2sn^2(u)cn^2(u)dn^2(u) \\
 & -2k^2sn(u)cn(v)dn(v)sn(v)cn(u)dn(u) - k^2sn^2(v)cn^2(u)dn^2(u) \\
= & 1-2k^2sn^2(u)sn^2(v)+k^4sn^4(u)sn^4(v)-k^2sn^2(u)(1-sn^2(v))(1-k^2sn^2(v)) \\
 & -k^2sn^2(v)(1-sn^2(u))(1-k^2sn^2(u)) - 2k^2sn(u)cn(v)dn(v)sn(v)cn(u)dn(u) \\
= & 1-2k^2sn^2(u)sn^2(v)+k^4sn^4(u)sn^4(v)-k^2sn^2(u)+k^2sn^2(u)sn^2(v) \\
 & +k^4sn^2(v)sn^2(u)-k^4sn^4(v)sn^2(u)-k^4sn^2(v)+k^2sn^2(v)sn^2(u) \\
 & +k^4sn^2(u)sn^2(v)-k^4sn^4(u)sn^2(v)-sk^2sn(u)cn(v)dn(v)sn(v)sn(u)cn(u)dn(u) \\
= & (1-k^2sn^2(u))(1-k^2sn^2(v))+k^4sn^2(v)sn^2(u)-k^4sn^4(v)sn^2(u) \\
 & +k^4sn^2(u)sn^2(v)-k^4sn^4(u)sn^2(v)-2k^2sn(u)cn(v)dn(v)sn(v)cn(u)dn(u) \\
= & (1-k^2sn^2(u))(1-k^2sn^2(v))+k^4sn^2(u)sn^2(v)(1-sn^2(u))(1-sn^2(v)) \\
 & -2k^2sn(u)cn(v)dn(v)sn(v)cn(u)dn(u) \\
= & dn(u)dn(v)+k^4sn^2(u)sn^2(v)cn^2(u)cn^2(v) \\
 & -2k^3sn^2(u)cn^2(v)dn^2(v)sn^2(v)cn^2(u)dn^2(u) \\
= & (dn(u)dn(v)-k^2sn(u)sn(v)cn(u)cn(v))^2 \\
\Rightarrow & dn(u+v) = \pm\frac{dn(u)dn(v)-k^2sn(u)sn(v)cn(u)cn(v)}{1-k^2sn^2(u)sn^2(v)}
\end{array}$$

Let $u = 0$

$$\begin{aligned}
dn(v) &= \pm\frac{dn(0)dn(v)-k^2sn(0)sm(v)cn(0)cn(v)}{1-k^2sn^2(0)sn^2(v)} \\
&= \pm dn(v)
\end{aligned}$$

So again take the positive term.
Hence $dn(u+v) = \frac{dn(u)dn(v)-k^2sn(u)sn(v)cn(u)cn(v)}{1-k^2sn^2(u)sn^2(v)}$

■

This concludes the theory on Jacobian Elliptic Functions. In the next section, I will rigourously define elliptic functions and generalise the theory. In this section I will prove some of the main results in the general theory of elliptic functions. The first thing I need to do is define an elliptic function.

DEFINITION

A function, f, which is holomorphic except at poles and has singularities in a finite part of the plane and satisfies the following equations:

$$\begin{aligned} f(z+\omega_1) &= f(z) \\ \text{and } f(z+\omega_2) &= f(z) \text{ for some } \omega_1 \text{ and } \omega_2 \in \mathbb{C} \end{aligned}$$

is called elliptic. ■

Firstly I shall prove some important theorems in the theory of elliptic functions and then I will draw some conclusions on the nature of elliptic functions based on these results before looking at some specific examples.

LEMMA 2

An elliptic function without poles is constant

Proof. Let $f(z)$ be an elliptic function with periods ω_1 and ω_2. $f(z)$ can be enclosed by a parallelogram, P_α, with vertices $\alpha, \alpha+\omega_1, \alpha+\omega_2$ and $\alpha+\omega_1+\omega_2$. This idea and why it works will be discussed later. As we have no poles and f is bounded, Liouville's Theorem says that f is constant. ■

LEMMA 3

The sum of the residues of an elliptic function is zero.

Proof. Let P_α be the same as in the previous proof. By the Residue Theorem we have

$$\begin{aligned} \textstyle\sum_{j=1}^{n}(f, z_j) &= \tfrac{1}{2\pi i}\int_{\partial P_\alpha} f(z)dz \text{ where there is a pole at every } z_j \\ &= \tfrac{1}{2\pi i}\left(\int_{\alpha}^{\alpha+\omega_1} f(z)dz + \int_{\alpha+\omega_1}^{\alpha+\omega_1+\omega_2} f(z)dz + \int_{\alpha+\omega_1+\omega_2}^{\alpha+\omega_2} f(z)dz + \int_{\alpha+\omega_2}^{\alpha} f(z)dz \right) \end{aligned}$$

As f is periodic $f(z) = f(z+\omega_1)$ and $f(z) = f(z+\omega_2)$, so changing the variables on the second and third integral gives

$$\begin{aligned} \textstyle\sum_{j=1}^{n}(f, z_j) &= \tfrac{1}{2\pi i}\left(\int_{\alpha}^{\alpha+\omega_1} + \int_{\alpha}^{\alpha+\omega_2} f(z+\omega_1)dz + \int_{\alpha+\omega_1}^{\alpha} f(z+\omega_2)dz + \int_{\alpha+\omega_2}^{\alpha} f(z)dz \right) \\ &= \tfrac{1}{2\pi i}\int_{\alpha}^{\alpha+\omega_1} f(z) - f(z+\omega_2)dz - \tfrac{1}{2\pi i}\int_{\alpha}^{\alpha+\omega_2} f(z) - f(z+\omega_1)dz \end{aligned}$$

As f is periodic both of these integrals are zero thus $\sum_{j=1}^{n}(f, z_j) = 0$ ■

Now I can define the order of an elliptic function. Let c be a constant. The order of f is the number of roots of the equation $f(z) = c$ which is equal to the number of poles in P_α.

THEOREM 4

*E*lliptic functions have order ≥ 2

Proof. If an elliptic function has order 1, it must have a non-zero residue. However this contradicts lemma 3. So elliptic functions must have order > 1. ■

In this essay I will just examine the case where the order of the elliptic function is 2. When the order is two, there are two subcases. One when there is a single irreducible double pole with zero residue and another when there are two poles with equal but opposite residues.

I have already looked at the elliptic functions with two poles, as these are the Jacobian Elliptic Functions. In the next section, I will consider the other type of order 2.

4 Weierstraß-$\wp$ Function

In this section, I will start off by constructing an elliptic function of order two with a single irreducible pole with zero residue.

Firstly, I will define the function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \text{ where } \omega = n_1\omega_1 + n_2\omega_2$$

This is the Weierstraß-$\wp$ function.

This function has its pole at the origin.

THEOREM 5

*T*he above sum converges, has a double period and hence is an elliptic function.

Proof. This argument is taken from [6] P.433-435.

Firstly estimate,

$$\begin{aligned} \left|\tfrac{1}{z-\omega)^2} - \tfrac{1}{\omega^2}\right| &= \left|\tfrac{\omega^2-(z-\omega)^2}{\omega^2(z-\omega)^2}\right| \\ &= \left|\tfrac{\omega^2-z^2+2\omega z-\omega^2}{\omega^2(z-\omega)^2}\right| \\ &= \left|\tfrac{z(2\omega-z)}{\omega^2(z-\omega)^2}\right| \end{aligned}$$

Let $|\omega| > 2|z|$ so

$$\begin{aligned} \left|\tfrac{2\omega z-z^2}{\omega^2 z^2-2\omega^3 z+\omega^4}\right| &\leq \tfrac{2|\omega z|-|z^2|}{|\omega^2 z^2|-2|\omega^3 z|+|\omega^4|} \text{ by triangle inequality} \\ &\leq \tfrac{4|z^2|-|z^2|}{|\omega^2 \frac{|\omega^2|}{4}|-2|\omega^3|\frac{|\omega|}{2}+|\omega^4|} \\ &\leq \tfrac{12|z^2|}{|\omega^4|} \\ &\leq \left|\tfrac{z}{\omega}\right|\tfrac{2|z|}{|\omega^3|} \\ &\leq \tfrac{1}{2}\tfrac{12|z|}{|\omega^3|} \\ &\leq \tfrac{6|z|}{|\omega^3|} \end{aligned}$$

So the series converges if $\sum_{\omega\neq 0}\frac{6|z|}{|\omega^3|}$ converges.

This is true since $\frac{\omega_2}{\omega_1}$ is non-real, so $\exists K > 0$ such that $|n_1\omega_1 + n_2\omega_2| \geq K(|n_1| + |n_2|)\forall(n_1, n_2) \in \mathbb{R}$.

Consider only integer pairs of (n_1, n_2). There are $4n$ which have the property $|n_1| + |n_2| = n$.

So $\sum_{\omega\neq 0}|\frac{1}{\omega^3}| \leq 4\frac{1}{k^3}\sum_1^\infty \frac{1}{n^3}$ which converges.

Now al that is left to prove that φ is an elliptic function is to prove that it is doubly periodic.

Let $f(z) = \frac{1}{z^2} + \sum_{\omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$

$$f'(z) = \frac{-2}{z^3} - \sum_{\omega\neq 0}\frac{2}{(z-omega)^3}$$

then

$$f'(z) = -2\sum_{\omega}\frac{1}{(z-\omega)^3}$$

This sum has two distinct periods, ω_1, ω_2, which implies that $f(z+\omega_1) - f(z) =$ a constant and $f(z+\omega_2) - f(z) =$ a constant. As f is even by definition, let $z = -\frac{\omega_1}{2}$

$$\Rightarrow f(\omega_1 - \frac{\omega_1}{2}) - f(-\frac{\omega_1}{2}) = f(\frac{\omega_1}{2}) - f(\frac{\omega_1}{2}) = 0$$

and now let $z = -\frac{\omega_2}{2}$

$$\Rightarrow f(\omega_2 - \frac{\omega_2}{2}) - f(-\frac{\omega_2}{2}) = f(\frac{\omega_2}{2}) - f(\frac{\omega_2}{2}) = 0$$

From our definition of φ and f it is clear that $\varphi(z) - f(z)$ is a constant and by the construction of φ at the origin, $\varphi(z) = f(z)$ and so φ is an elliptic function. ■

φ has zero reside and so can be integrated to obtain $F(z) = -\frac{1}{z} - \sum_{\omega \neq 0} \frac{1}{z-\omega} - \frac{1}{\omega} - \frac{z}{\omega^2}$ by convention $F(z)$ is called $-\zeta(z)$ so $\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}$

Once again convergence has to be shown. Firstly estimate

$$\begin{aligned} \left|\tfrac{1}{z-\omega} + \tfrac{1}{\omega} + \tfrac{z}{\omega^2}\right| &= \left|\tfrac{\omega^2+(z-\omega)\omega+z(z-\omega)}{\omega^2(z-\omega)}\right| \\ &= \left|\tfrac{\omega^2+z\omega-\omega^2+z^2-z\omega}{\omega^2 z-\omega^3}\right| \\ &= \left|\tfrac{z^2}{\omega^2 z-\omega^3}\right| \end{aligned}$$

Let $|\omega| > 2|z|$

$$\begin{aligned} \left|\tfrac{z^2}{\omega^2 z-\omega^3}\right| &\leq \tfrac{|z^2|}{|\omega^3 z|-|\omega^3|} \text{ by triangle inequality} \\ &\leq \left|\tfrac{z}{\omega}\right| \tfrac{|z|}{|\omega||\frac{\omega}{2}|-|\omega^2|} \\ &\leq \tfrac{|z|}{2(|\omega^2|\cdot\frac{1}{2}-|\omega^2|)} \\ &\leq \tfrac{|z|}{|\omega^2|} \end{aligned}$$

Now only have to show that $\sum_{\omega \neq 0} \frac{1}{\omega^2}$ converges.

This series converges since $\frac{\omega_2}{\omega_1}$ is non-real and so $\exists M > 0$ st $|n_1\omega_1 + n_2\omega_2| \geq M(|n_1| + |n_2|) \forall (n_1, n_2) \in \mathbb{R}$.

Hence ζ converges.

As φ is periodic it follows that ζ is periodic and thus $\zeta(z + \omega_1) = \zeta(z) + \eta_1$ (16) and $\zeta(z + \omega_2) = \zeta(z) + \eta_2$ (17).

There is a very important relationship between η_1, η_2, ω_1 and ω_2.

THEOREM 6

*T*he following relationship holds:

$$2\pi i = \omega_2 \eta_1 - \omega_1 \eta_2$$

Proof.

Recall the parallelogram, P_α, used earlier. Now use the Residue Theorem.

$$\begin{aligned} 1 = \textstyle\sum_{j=0}^{n}(f, z_j) &= \tfrac{1}{2\pi i} \textstyle\int_{\partial P_\alpha} \zeta(z)dz \\ &= \tfrac{1}{2\pi i}\left(\textstyle\int_{\alpha}^{\alpha+\omega_1} \zeta(z)dz + \int_{\alpha+\omega_1}^{\alpha+\omega_1+\omega_2} \zeta(z)dz + \int_{\alpha+\omega_1+\omega_2}^{\alpha+\omega_2} \zeta(z)dz + \int_{\alpha+\omega_2}^{\alpha} \zeta(z)dz \right) \end{aligned}$$

Use (16) and (17) to change the variables in the second and third integrals to get:

$$\begin{aligned} 2\pi i &= \textstyle\int_{\alpha}^{\alpha+\omega_1} \zeta(z)dz + \int_{\alpha}^{\alpha+\omega_2} \zeta(z+\omega_1) + \eta_1 dz + \int_{\alpha+\omega_1}^{\alpha} \zeta(z+\omega_1) + \eta_2 dz + \int_{\omega_2}^{\alpha} \zeta(z)dz \\ &= \textstyle\int_{\alpha}^{\alpha+\omega_1} \zeta(z) - \zeta(z+\omega_2)dz - \int_{\alpha}^{\alpha+\omega_2} \zeta(z) - \zeta(z-\omega_1)dz + \int_{\alpha}^{\alpha+\omega_1} \eta_1 dz + \int_{\alpha+\omega_1}^{\alpha} \eta_2 dz \\ &= (\alpha\eta_1 + \omega_2\eta_1 - \alpha\eta_1) + (\alpha\eta_2 - \alpha\eta_2 - \omega_1\eta_2) \\ \Rightarrow 2\pi i &= \omega_2\eta_1 - \omega_1\eta_2 \end{aligned}$$

■

This is called Legendre's Relation.

From the study of the Jacobian Elliptic Functions it seems sensible to assume that φ has an addition formula and here it is:

THEOREM 7

*W*e have

$$\varphi(z+y) = \frac{1}{4}\left(\frac{\varphi'(z) - \varphi'(y)}{\varphi(z) - \varphi(y)}\right)^2 - \varphi(z) - \varphi(y)$$

Proof.

It is important to note that if $y = \pm z(\text{mod } \omega_1.\omega_2)$ then the formula does not hold.

Let $y \neq \pm z(\text{mod } \omega_1, \omega_2)$.

The following equations are needed:

$$\begin{aligned} \varphi'(z) &= \alpha\varphi(z) + \beta \quad (18) \\ \varphi'(y) &= \alpha\varphi(y) + \beta \quad (19) \end{aligned}$$

Firstly look at $\varphi'(\gamma) - \alpha\varphi(\gamma) - \beta$. It has a triple pole when $\gamma = 0$ and so has three irreducible zeros by Theorem 2 and hence $\gamma = z, \gamma = y$ and $\gamma = -z - y$ are the poles.

So $\varphi(z), \varphi(y)$ and $\varphi(z+y)$ are the roots of the following cubic

$$4\varphi^3(\gamma) - \alpha^2\varphi^2(\gamma) - (2\alpha\beta + g_2)\varphi(\gamma) - (\beta^2 + g_2) = 0$$

From the theory of polynomials, it follows that $\varphi(z) + \varphi(y) + \varphi(z+y) = \frac{1}{4}\alpha^4$

Solving (18) and (19) for α gives $\alpha = \frac{\varphi'(z) - \varphi'(y)}{\varphi(z) - \varphi(y)}$

Hence $\varphi(z+y) = \frac{1}{4}\left(\frac{\varphi'(z)-\varphi'(y)}{\varphi(z)-\varphi(y)}\right)^2 - \varphi(z) - \varphi(y)$ ■

This has proved the addition formula however as noted earlier that it does not hold if $y = \pm z (\text{mod } \omega_1, \omega_2)$. It is possible to derive the formula when $y = z$, by taking the limit as y goes to z in the addition formula. Here is the necessary theorem.

THEOREM 8

*W*e have

$$\varphi(2z) = \frac{1}{4}\left(\frac{\varphi''(z)}{\varphi'(z)}\right)^2 - 2\varphi(z)$$

Proof.

Taking limits gives $\lim_{y\to z}\varphi(z+y) = \frac{1}{4}\lim_{y\to z}\left(\frac{\varphi'(z)-\varphi'(y)}{\varphi(z)-\varphi(y)}\right)^2 - \varphi(z) - \lim_{y\to z}\varphi(y)$

If $2z$ is not a period then we get $\varphi(2z) = \frac{1}{4}\lim_{h\to 0}\left(\frac{\varphi'(z)-\varphi'(z+h)}{\varphi(z)-\varphi(z+h)}\right) - 2\varphi(z)$

Apply Taylor's Theorem to get $\varphi(z+h)$ and $\varphi'(z+h)$ to get $\varphi(2z) = \frac{1}{4}\left(\frac{\varphi''(z)}{\varphi'(z)}\right) - 2\varphi(z)$ ■

This has shown some of the properties of φ.

This concludes the work on elliptic functions in my essay, in the next section I will consider some more abstract theory.

5 Algebraic Theory of Elliptic Functions

I am now going to go back and say something about the parallelogram, P_α, discussed earlier and comment on the field of elliptic functions.

P_α was formed by taking a point in the complex plane and using the periods of the elliptic function to calculate the sides of the parallelogram.

If this parallelogram is translated back to the origin, and then copied and placed end to end it can cover the whole of the complex plane. This is a lattice, $\mathbb{L} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, where in each parallelogram the behaviour of the elliptic function is identical due to its periodicity. Now quotient $\mathbb{C}$ by $\mathbb{L}$ to obtain a parallelogram with $\alpha = 0$ but this is on the complex torus. It can be proved that all elliptic functions belong to the field of functions of rational character over the complex torus, i.e. as all elliptic functions are quotients of the form $\frac{\mathbb{C}}{\mathbb{L}}$ then it must be rational. This gives some idea of how to look at elliptic functions algebraically.

6 Conclusion

To conclude I have given some motivation for the study of elliptic functions by looking at an example in classical mechanics which gives rise to them. There are of course many other systems which give rise to elliptic functions, but there is not enough space in this essay to include them all. I then looked briefly at the Jacobian Elliptic Functions before looking at the general theory of elliptic functions before analysing the Weierstraß-φ function and proving some of its basic properties. I then finally considered some of the more abstract theory.

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