A Proof of the De Rham Theorem using Induction on Open Sets

Mark Price

Abstract

An exposition of singular cohomology theory and De Rham cohomology leading up to the proof of the De Rham theorem using induction on open sets.

Contents

1 Introduction

The aim of this project is to prove the De Rham theorem using induction on open sets. The first few sections are mainly technical results which will be required in later proofs, while the last few sections are introductions to singular cohomology and De Rham cohomology with the final section being a proof of the De Rham theorem itself.

The first section shows the existence of a partition of unity. I have included this section rather than reference it as the style of proof used here will also be used later on. The second section covers induction on open sets. This is a little known result and is crucial to my proof of the De Rham theorem. The third section is on exact sequences. These appear in all areas of cohomology theory and so are worth their own section. This section also contains a statement and proof of the five lemma which is an important result in the theory of exact sequences and will be used in my proof of the De Rham theorem. The next section is an introduction to singular cohomology. It goes through the basic definitions and has two key results in it. The first being the existence of the Mayer-Vietoris sequence for singular cohomology and that the sequence gives the calculation of the singular cohomology of $\mathbb{R}^n$. The next section is an introduction to differential forms on manifolds and culminates in a proof of Stokes's theorem. Differential forms are the main elements of De Rham cohomology and Stokes's theorem is used in my proof of the De Rham theorem. The next section uses the ideas of the previous section to develop De Rham cohomology. Again the important results from this chapter are the existence of Mayer-Vietoris sequences for De Rham cohomology and the De Rham cohomology of $\mathbb{R}^n$. The final section is a proof of the De Rham theorem using induction on open sets.

2 Partition of Unity

The main purpose of this section is to define a partition of unity and show that one exists on any open set of $\mathbb{R}^n$ with an open cover. This will be done through a series of lemmas which will be mostly showing that there exist functions with the certain properties which can be combined to get a partition of unity. The books used in this section are[4], [6], [7], [10], [11] and [13]. Firstly, I want to define the support of a function.

Definition 2.1 (Support of a Function). Let $f\colon U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$. The support of f is the set $supp_U\ (f) = \overline{\{x \in U | f(x) \neq 0\}}$.

The support of a function picks out the area where it is non-zero.

The next lemma and its corollaries just show the existence of functions which will be required in the proof of a partition of unity.

Lemma 2.2. *The function* $\omega\colon \mathbb{R} \to \mathbb{R}$ *defined by* $\omega(t) = \begin{cases} exp\ \left(-\frac{1}{t}\right), & if\ t > 0 \\ 0, & if\ t \leq 0 \end{cases}$ *issmooth*

Proof. Only need to check smoothness at $t = 0$ and it is enough to show that

$$lim_{t \to 0^+} \frac{\omega^{(n-1)}(t)}{t} = 0\ \forall\ n \geq 1$$

Differentiating n times gives the existence of polynominals $p_0, p_1, p_2, \ldots$ which satisfy $\omega^{(n)}(t) = p_n\left(\frac{1}{t}\right)\ exp\ \left(-\frac{1}{t}\right)$ for $t > 0$ and $n \geq 0$. We now have

$$lim_{t \to 0^+}\left(\left(\frac{1}{t}\right)^k\ exp\ \left(-\frac{1}{t}\right)\right) = \ lim_{x \to \infty} \frac{x^k}{exp\ (x)} = 0$$

□

Corollary 2.3. *For real numbers* $a < b$ *there exists a smooth function* $\psi\colon \mathbb{R} \to [0,1]$ *such that* $\psi(t) = 0$ *for* $t \leq a$ *and* $\psi(t) = 1$ *for* $t \geq b$.

Proof. Set $\psi(t) = \frac{\omega(t-a)}{\omega(t-a)+\omega(b-t)}$ □

Corollary 2.4. *For* $x \in \mathbb{R}^n$ *and* $\epsilon > 0$ *there exists a smooth function* $\phi\colon \mathbb{R}^n \to [0,\infty)$ *such that* $D_\epsilon(x) = \phi^{-1}((0,\infty))$ *where* $D_\epsilon(x) = \{y \in \mathbb{R}^n |\ \|y - x\| \leq \epsilon\}$, *ie the disc of radius* ϵ *centred at* x.

Proof. Define ϕ by the formula

$$\phi(y) = \omega\left(\epsilon^2 - \|y - x\|^2\right) = \omega\left(\epsilon^2 - \sum_{j=1}^{n}(y_j - x_j)^2\right)$$

□

Now we have that the support of ϕ is the closed disc of radius ϵ centred at x, ie

$$supp_U\ \phi = \overline{D_\epsilon}(x)$$

The next few lemmas look at the properties of open sets required for the proof of the existence of a partition of unity.

Lemma 2.5. *An arbitrary open set $U \subset \mathbb{R}^n$ can be written in the form $U = \bigcup_{m=1}^{\infty} K_m$, where the K_m are compact and $K_m \subset K_{m+1}^{\circ}$ for $m \geq 1$. K° means the interior of the set K.*

Proof. Let $K_m = \overline{D_{2m}}(0) \setminus \bigcup_{x \in \mathbb{R}^n \setminus U} D_{\frac{1}{2}m}(c)$. This satisfies the necessary conditions. □

Lemma 2.6. *For an arbitrary open set $U \subseteq \mathbb{R}^n$ and a cover $\mathcal{V} = (V_i)_{i \in I}$ of U by open sets, it is possible to find a sequence (x_j) in U and a sequence (ϵ_j) of positive real numbers that satisfy the following conditions :*

(*a*) $U = \bigcup_{j=1}^{\infty} D_{\epsilon_j}(x_j)$

(*b*) *For every j there exists $i(j) \in I$ with $D_{2\epsilon_j}(x_j) \subseteq V_{i(j)}$*

(*c*) *Every $x \in U$ has a neighbourhood W that intersects only finitely many of the balls $D_{2\epsilon_j}(x_j)$*

Proof. Choose $K_m\,(m \geq 1)$ as in lemma 2.5. Set $K_0 = K_{-1} = \emptyset$. For $m \geq 1$ let $B_m = K_m \setminus K_{m-1}^{\circ}$, $U_m = K_{m+1}^{\circ} \setminus K_{m-2}$. For every $x \in B_m$ we can find $\epsilon(x) > 0$ such that $D_{2\epsilon(x)}$ is contained in both U_m and at least one of the sets V_i. The Heine-Borel property for B_m ensures the existence of $X_{m,j} \in B_m$ and $\epsilon_{m,j} > 0\,(1 \leq j \leq d_m)$ such that

(*i*) $B_m \subseteq \bigcup_{j=1}^{d_m} D_{\epsilon_{m,j}}(x_{m,j})$

(*ii*) Every $D_{2\epsilon_{m,j}}(x_{m,j})$ is contained in U_m and at least on of the sets V_i

Now re-index the families $(x_{m,j})$ and $(\epsilon_{m,j})$, where $m \geq 1$ and $1 \leq j \leq d_m$. From (i) and (ii) we get

$$U = \bigcup_{m=1}^{\infty} B_m \subseteq \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{r_m} D_{\epsilon_{m,j}}(x_{m,j}) \subset \bigcup_{m=1}^{\infty} U_m \subset U$$

which proves (a). (b) follows straight from (ii). To prove (iii) for every $x \in U$ choose $m_0 \geq 1$ with $x \in U_{m_0}$. Since $U_m \cap U_{m_0} = \emptyset$ when $m \geq m_0 + \zeta$, it follows that U_{m_0} can intersect $D_{\epsilon}(x_{m,j})$ only when $m \leq m_0 + 2$. □

We now have enough information to prove the existence of a partition of unity.

Theorem 2.7 (Existence of a Partition of Unity). *If $U \subseteq \mathbb{R}^n$ is open and $\mathcal{V} = (V_i)_{i \in I}$ is a cover of U by open sets then there exist smooth functions $\phi_i \colon U \to [0,1]$ satisfying*

(*i*) *$supp_U(\phi_1) \subseteq V_i$ for all $i \in I$*

(*ii*) *Every $x \in U$ has a neighbourhood W on which only finitely many ϕ_i do not vanish*

(*iii*) *For every $x \in U$, $\sum_{i \in I} \phi_i(x) = 1$*

ϕ_i is a partition of unity which is subordinate to the cover $\mathcal{V}$.

Proof. Choose (x_j), (ϵ_j) and $i(j) \in I$ as in lemma 2.6. Apply corollary 2.4 to find smooth functions $\psi_j \colon \mathbb{R}^n \to [0, \infty)$ with $D_{\epsilon_j}(x_j) = \psi^{-1}((0,\infty))$. Condition (c) from lemma 2.6 guarantees that the function $\psi \colon U \to \mathbb{R}$ given by $\psi(x) = \sum_j \psi_j(x)$ is smooth because the sum is finite on a neighbourhood of any $x \in U$. From (a) of lemma 2.6 we have that $\psi(x) > 0$ for all $x \in U$.

Let $\tilde{\psi_j} \colon U \to [0,\infty)$ be given by $\tilde{\psi_j}(x) = \psi_j(x)\,\psi^{-1}(x)$. These are smooth with $D_{\epsilon_j}(x_j) = \tilde{\psi_j}^{-1}((0,\infty))$ and $\sum_j \tilde{\psi_j}(x) = 1$ for all $x \in U$. Set $\phi_j = \sum \tilde{\psi_j}$, for $i \in I$, summed over the set J_i of indices j for which $i(j) = i$ (in particular $\phi_i = 0$ when $J_i = \emptyset$). By lemma 2.6 (c), it follows that ϕ_i is smooth on U. These functions satisfy (ii) and (iii). It just remains to show (i). Every $x \in supp_U(\phi_i)$ has a neighbourhood $W \subseteq U$ that satisfies lemma 2.6 (c). Thus the restriction of $\phi_i|_W$ becomes a sum of finitely many $\tilde{\psi_{j_\nu}}|_W$ with $j_\nu \in J_i$. There is at least one $j_\nu \in J_i$, with $x \in supp_U\left(\tilde{\psi_{j_\nu}}\right) = \overline{D_{\epsilon_{j_\nu}}}(x_{j_\nu}) \subseteq D_{2\epsilon_{j_\nu}}(x_{j_\nu}) \subseteq V_i$ we have $x \in V_i$. □

3 Induction on Open Sets

In this section I want to prove a theorem on how induction can be applied to open sets. It will be the main result used in the proof of the De Rham theorem. The book used in this section is [7]

Theorem 3.1 (Induction on Open Sets). *Let M be a smooth n dimensional manifold equipped with an open cover $\mathcal{V} = (V_\beta)_{\beta \in B}$. Suppose $\mathcal{U}$ is a collection of open subsets of M that satisfy the following four conditions:*

- (i) $\emptyset \in \mathcal{U}$
- (ii) *Any open subset $U \subseteq V_\beta$ diffeomorphic with $\mathbb{R}^n$ belongs to $\mathcal{U}$*
- (iii) *If $U_1, U_2, U_1 \cap U_2$ belong to $\mathcal{U}$ then $U_1 \cup U_2 \in \mathcal{U}$*
- (iv) *If $U_1, U_2, \ldots$ is a sequence of pairwise disjoint open subsets with $U_i \in \mathcal{U}$ then their union $\bigcup_i U_i \in \mathcal{U}$*

Then M belongs to $\mathcal{U}$

The following lemma is required for the proof.

Lemma 3.2. *Given the conditions of theorem 3.1. Let $U_1, U_2, \ldots$ be a sequence of open subsets of M with compact closure, which satisfy:*

- (a) $\bigcap_{j \in J} U_j \in \mathcal{U}$ *for any finite subset J*
- (b) $(U_j)_{j \in \mathbb{N}}$ *is locally finite*

Then the union $\bigcup_{i=1}^n U_i$ belongs to $\mathcal{U}$.

Proof. Firstly need to show that $U_{j_1} \cup U_{j_2} \cup \ldots \cup U_{j_m} \in \mathcal{U}$ is true for every set of indices $j_1, \ldots j_m$. This is done by an induction argument on m. The cases $m = 1, 2$ follow from (a) and theorem 3.1.(iii). So let $m \geq 3$ and assume that the claim is true for sets of $m-1$ indices. Let $V = U_{j_2} \cup \ldots \cup U_{j_m}$ and $U_{j_1} \cap V = \bigcup_{\nu=2}^m U_{j1} \cap U_{j_\nu} \in \mathcal{U}$. Now apply the induction argument to the sequence $(U_{J_1} \cap U_j)_{j \in \mathbb{N}}$. This together with theorem 3.1.(iii) gives us $U_{j_1} \cup \ldots U_{j_m} \in \mathcal{U}$. By (a) we have $U_i \cap U_j \in \mathcal{U}$ and so

$$\bigcup_{\nu=1}^{m} (U_{i_\nu} \cap U_{j_\nu}) \in \mathcal{U} \tag{1}$$

for any set of $2m$ indices $i_1, j_1, \ldots, i_m, j_m$.

Inductively, define index sets I_m and open sets $W_m \subseteq M$ as follows:

$$\begin{aligned} I_1 &= 1, \\ W_1 &= U_1, \\ \text{and for } m \geq 2 \;\; I_m &= \{m\} \cup \{i | i > m,\; U_i \cap W_{m-1} \neq \emptyset\} \setminus \bigcup_{j=1}^{m-1} I_j \\ W_m &= \bigcup_{i \in I_m} U_i \end{aligned} \tag{2}$$

If I_{m-1} is finite, then W_{m-1} has compact closure and (b) implies that W_{m-1} only intersects finitely many of the sets U_i. This shows inductively that I_m is indeed finite for all m. Moreover, if $m \geq 2$ does not belong to any U_j with $j < m$ then by definition of I_m, I_j belongs to I_m. Hence $\mathbb{N}$ is the disjoint union of all the finite sets I_m.

All finite unions are in $\mathcal{U}$, so $W_m \in \mathcal{U}$. If $I_m = \emptyset$, $W_m = \emptyset \in \mathcal{U}$. Similarly, (1) shows that

$$W_m \cap W_{m+1} = \bigcup_{(i,j) \in I_m \times I_{m+1}} U_i \cap U_j \in \mathcal{U}$$

(2) implies that $W_m \cap W_k = \emptyset$ if $k \geq m+2$. If the intersection were non-empty, then $\exists i \in I_k$ with $W_m \cap U_i \neq \emptyset$, and by (2) $i \in I_j$ for some $j \leq m+1$.

By theorem 3.1.(iv), the following sets are in $\mathcal{U}$:

$$W^{(0)} = \bigcup_{j=1}^{\infty} W_{2_j},\ W^{(1)} = \bigcup_{j=1}^{\infty} W_{2_{j-1}},\ W^{(0)} \cap W^{(1)} = \bigcup_{j=1}^{\infty} (W_m \cap W_{m+1})$$

By theorem 3.1.(iii) , $\cup_{i=1}^{\infty} U_i = W^{(0)} \bigcup W^{(1)} \in \mathcal{U}$ □

Proof. (of theorem 3.1)

Consider the special case where $M = W \subset \mathbb{R}^n$ is an open subset. Here we use the maximum norm on $\mathbb{R}^n$,

$$\|x\|_\infty = max_{1 \leq i \leq n} |x_i|$$

whose open balls are the cubes $\prod_{i=1}^{n} (a_i, b_i)$. Recall lemma 2.6, if the proof is repeated with the norm being replaced by the maximum norm and $D_{\epsilon_j}(x_j)$ becomes $\| \ \|_\infty$-balls U_j, we find that we have a sequence of U_j which satisfies:

(i) $W = \bigcup_{j=1}^{\infty} U_j = \bigcup_{j=1}^{\infty} \overline{U_j}$

(ii) $(U_j)_{j \in \mathbb{N}}$ is locally finite

(iii) Each U_j is contained in at least one V_β

A finite intersection $U_{j_1} \cap \ldots U_{j_m}$ is either empty or of the form $\prod_{i=1}^{n} (a_i, b_i)$ which is diffeomorphic to $\mathbb{R}^n$. Thus by lemma 3.2 $W \in \mathcal{U}$.

In the general case, firstly consider an open coordinate patch (U, h) of M, $h \colon U \to W$ a diffeomorphism onto an open subset $W \subseteq \mathbb{R}^n$. W is covered by $(h^{-1}(V_\beta))_{\beta \in B}$ and let $\mathcal{U}^h$ be the open sets of W whose images by h^{-1} belong to the given $\mathcal{U}$. Thus by lemma 3.2 $U \in \mathcal{U}$ for each coordinate patch.

If M is compact then cover M by a finite cover of coordinate patches and apply the above argument to see that $M \in \mathcal{U}$.

If M is non-compact use a sequence of coordinate patches with compact closure instead of $D_{\epsilon_j}(x_j)$ in the proof of lemma 2.6 to get a locally finite cover of M □

This is a very powerful theorem as will be shown in my proof of the De Rham theorem.

4 Exact Sequences

Exact sequences arise in all forms of cohomology so I want to cover them now before moving onto singular cohomology and De Rham cohomology. The books used in this section are [5], [7], [8],[9] and [12].

Firstly I want to define an exact sequence.

Definition 4.1 (Exact Sequence). A sequence of Abelian groups and homomorphisms

$$\cdots \longrightarrow A_{n-1} \xrightarrow{h_{n-1}} A_n \xrightarrow{h_n} A_{n+1} \xrightarrow{h_{n+1}} \cdots$$

is called exact if the image of each homomorphism is the kernel of the following homomorphism, ie $Im\,(h_{n-1}) = Ker\,(h_n)$.

An important result in the theory of exact sequences is the five lemma.

Theorem 4.2 (The Five Lemma). *In the following commutative diagram of Abelian groups, if the rows are exact and α, β, δ, ϵ are isomorphisms then so is γ.*

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

Proof. It is sufficient to show:

(i) γ is injective

(ii) γ is surjective

(i) Let $c \in C$ and $\gamma(c) = 0$. Then $\delta((c)) = k'(\gamma(c)) = 0$. Hence $k(c) = 0$ since δ is an isomorphism. So $c \in Ker\,(k) = Im\,(j)$, choose b such that $j(b) = c$. Then $j'(\beta(b)) = \gamma(j(b)) = \gamma(c) = 0$. So $\beta(b) \in Ker\,(j') = Im\,(i')$; choose a' such that $i'(a') = \beta(b)$. α is an isomorphism so choose a such that $\alpha(a) = a'$. Then $\beta(i(a)) = i'(\alpha(a)) = i'(a') = \beta(b)$. β is an isomorphism, so $i(a) = b$. Therefore $0 = j(i(a)) = j(b) = c$. Thus $c = 0$. Hence γ is injective.

(ii) Let $c' \in C'$. Then $k'(c') = \delta(d)$ for some $d \in D$ since δ is surjective. Since ϵ is injective and $\epsilon(l(d)) = l'(\delta(d)) = l'(k'(c')) = 0$, we get $l(d) = 0$. Hence $d = k(c)$ for some $c \in C$ by exactness of top row. The difference $c' - \gamma(c)$ maps to 0 under k' since $k'(c') - k'(\gamma(c)) = k'(c') - \delta(k(c)) = k'(c') - \delta(d) = 0$. Therefore $c' - \gamma(c) = j'(b')$ for some $b' = \beta(b)$ for some $b \in B$, and then $\gamma(c + j(b)) = \gamma(c) + \gamma(j(b)) = \gamma(c) + j'(\beta(b)) = \gamma(c) + j'(b') = c'$, showing that γ is surjective.

□

This is very powerful as it allows maps to be shown to be isomorphisms without having to do any explicit calculation. I now want to define a short exact sequence.

Definition 4.3 (Short Exact Sequence). A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Theorem 4.4.

(i)

$$0 \xrightarrow{\alpha} A \xrightarrow{\beta} B$$

is exact iff $Ker\,(\alpha) = 0$, *ie* α *is injective*

(ii)

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} 0$$

is exact iff $Im\,(\alpha) = B$, *ie* α *is surjective*

(iii)

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is exact iff α *is an isomorphism*

(iv)

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact iff α *is injective,* β *is surjective and* $Ker\,(\beta) = Im\,(\alpha)$*. So* β *induces an isomorphism* $C \cong \frac{B}{Im(\alpha)}$

(v)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of vector spaces. Then B *is finite dimensional if both* A *and* C *are, and* $B \cong A \oplus C$

Proof.

(i) $(\Rightarrow)$Exactness at A gives $Ker\,(\beta) = Im\,(\alpha) = 0$
$(\Leftarrow)$$Ker\,(\beta) = 0$ but $Im\,(\alpha) = 0$, so $Ker\,(\beta) = Im\,(\alpha)$, so exact

(ii) $(\Rightarrow)$Exactness at B gives $Im\,(\alpha) = Ker\,(\beta) = B$
$(\Leftarrow) Im\,(\alpha) = B$ but $Ker\,(\beta) = B$ so $Im\,(\alpha) = Ker\,(\beta)$, so exact

(iii) Follows straight from (i) and (ii)

(iv) The iff part follows straight from (i) and (ii). The first isomorphism theorem tells us that $c \cong \frac{B}{Ker(\beta)}$ but by exactness $Ker\,(\beta) = Im\,(\alpha)$, thus $C \cong \frac{B}{Im(\alpha)}$

(v) Let $\{a_i\}$ be a basis of A and $\{c_j\}$ be a basis of C. g is surjective by (iv) and so there exists $b_j \in B$ with $g\,(b_j) = c_j$. Then $\{f\,(a_i)\,, b_j\}$ is a candidate for a basis for B. For $b \in B$ we have $g\,(b_j) = \sum \lambda_j c_j$. Thus $b - \sum \lambda_j b_j \in Ker\,(g)$. Since the sequence is exact $Ker\,(g) = Im\,(f)$, so $b - \sum \lambda_j b_j = f\,(a)$, so

$$b - \lambda_j b_j = f\left(\sum \mu_i a_i\right) = \sum \mu_i f\,(a_i)$$

Thus b can be rewritten as a linear combination of $\{b_j\}$ and $\{f\,(a_i)\}$. It remains to show that $\{b_j, f\,(a_i)\}$ are linearly independent. b_j are linearly independent as they form a basis of B. $f\,(a_i)$ are linearly independent since f is injective and they form a basis of A. Thus $\{b_j, f\,(a_i)\}$ is linearly independent. Hence $\{b_j, f\,(a_i)\}$ forms a basis of B and so $B \cong A \oplus C$.

□

Here are some examples of how to use short exact sequences to calculate groups:

Example 4.5. *Let*

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\;f\;} A \xrightarrow{\;g\;} \mathbb{Z}_2 \longrightarrow 0$$

be exact. What is A?

Case 1 $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$f \colon \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ $f\,(1) = (0, 1)$
$g \colon \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ $g\,(a, b) = b$
$Ker\,(g) = \{(a, 0)\,|a \in \mathbb{Z}_2\} = \{(0, 0) + (1, 0)\} = Im\,(f)$ So exact

Case 2 $a = \mathbb{Z}_4$
$f \colon \mathbb{Z}_2 \to \mathbb{Z}_4$ $f\,(1) = 2$
$g \colon \mathbb{Z}_4 \to \mathbb{Z}_2$ $g\,(1) = 1$
$Im\,(f) = \{0, 2\} = Ker\,(g)$
So exact

Example 4.6. *Let*

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\;f\;} A \xrightarrow{\;g\;} \mathbb{Z}_3 \longrightarrow 0$$

be exact. What is A?

$A = \mathbb{Z}_6 = \mathbb{Z}_3 \oplus \mathbb{Z}_2$ by theorem 4.4.(v)

5 Singular Cohomology

In this section I want to cover enough singular cohomology to be able to prove the De Rham theorem. This means I have to calculate the singular cohomology of $\mathbb{R}^n$ and show the existence of Mayer-Vietoris sequences for singular cohomology. The books used in this section were [3], [8], [9] and [12].

Throughout this section let X be a topological space and G and Abelian group. We define the following:

Definition 5.1 (The group of singular n-chains with coefficients in G). Let $C^n(X;G) = Hom(C_n(X), G)$. Here $C_n(X)$ is the group of singular chains, which comes from homology theory and $Hom(A,B)$ is the set of all homomorphisms from A to B. Then $C^n(X;G)$ is the group of singular n-chains with coefficients in G.

Definition 5.2 (n-cochain). $\varphi \in C^n(X;G)$ is an n-cochain and it assigns to each singular n-simplex $\sigma\colon \Delta^{n+1} \to X$ a value $\varphi(\sigma) \in G$.

Definition 5.3 (Coboundary map). The map $\delta_n\colon C^n(X;G) \to C^{n+1}(X;G)$ is the coboundary map. δ_n acts on an n-chain in the following way:

$$\delta_n \varphi(\sigma) = \sum_i (-1)\, \varphi(\sigma|\, [\nu_0, \ldots, \widehat{\nu_i}, \ldots, \nu_{n+1}])$$

The following property of the coboundary map is important:

Proposition 5.4. $\delta_{n+1} \circ \delta_n = 0$

Proof.

$$\begin{aligned}(\delta_{n+1} \circ \delta_n)\, \varphi(\sigma) &= \sum_{j<i} (-1)^i (-1)^j \varphi(\sigma|\, [\nu_0, \ldots, \widehat{\nu_j}, \ldots, \widehat{\nu_i}, \ldots, \nu_{n+1}]) \\ &+ \sum_{i<j} (-1)^i (-1)^{j-1} \varphi(\sigma|\, [\nu_0, \ldots, \widehat{\nu_i}, \ldots, \widehat{\nu_j}, \ldots, \nu_{n+1}])\end{aligned}$$

If you swap i and j in the second sum, it is equal to the first but with the opposite sign. Thus they cancel. □

Now we can define a cochain complex, cocycles and cochains.

Definition 5.5 (Cochain complex). A cochain complex is a sequence of Abelian groups and homomorphisms

$$\ldots \xrightarrow{\delta_{n-1}} C^n(x;G) \xrightarrow{\delta_n} C^{n+1}(X;G) \xrightarrow{\delta_{n+1}} \ldots$$

where the composition $\delta_{n+1} \circ \delta_n = 0$ for each n. It is usual to denote the cochain complex by

$$C^* = \{\ldots \xrightarrow{\delta_{n-1}} C^n(x;G) \xrightarrow{\delta_n} C^{n+1}(X;G) \xrightarrow{\delta_{n+1}} \ldots\}$$

Definition 5.6 (Cocycles and Coboundaries). A cocycle is any element of $Ker(\delta)$. A coboundary is any element of $Im(\delta)$.

It is now possible to define the n$^{\text{th}}$ cohomology group.

Definition 5.7 (n$^{\text{th}}$ cohomology group of X with coefficients in G).

$$H^n(X;G) = \frac{Ker(\delta)}{Im(\delta)}$$

is the n$^{\text{th}}$ cohomology group with coefficients in G.

It is worth noting that $H^n(X;G)$ is an Abelian group. Now we need to extend this definition to cover pairs of maps (X,A) where $A \subset X$.

Definition 5.8.

$$H^n(X,A;G) = \frac{H^n(X;G)}{H^n(A;G)}$$

To be able to calculate the singular cohomology of $\mathbb{R}^n$ the following result is needed.

Lemma 5.9. *[Singular cohomology invariant under homotopy]If $f,g\colon (X,A) \to (Y,B)$ are homotopic as maps of pairs, then the induced homomorphisms*

$$f^*, g^*\colon H^n(Y,B;G) \to H^n(X,A;G)$$

are equal.

Proof. From homology theory we know that the homomorphisms

$$f_*, g_*\colon H_n(X,A) \to H_n(Y,B)$$

are equal. This then gives us our result by duality, ie reverse the arrows before taking homology groups, then take homology groups to ge the relevant result in cohomology. □

Now it is possible to calculate the singular cohomology of $\mathbb{R}^n$.

Proposition 5.10 (Singular cohomology of $\mathbb{R}^n$ with coefficients in $\mathbb{R}$).

$$H^n(\mathbb{R}^n;\mathbb{R}) = \begin{cases} \mathbb{R} & \textit{for } n=0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Firstly note that $\mathbb{R}^n$ is homotopic to a point. Thus by lemma 5.9 we only need to calculate the singular cohomology of a point. This can be done explicitly from the definitions. Firstly we have $C^n(pt) = Hom(C_n(X), \mathbb{R})$. Here

$$C_n(pt) = \begin{cases} 0 & \text{for } n>0 \\ \mathbb{R} & \text{for } n=0 \end{cases}$$

Thus for $n>0$, $H^n(pt;\mathbb{R}) = 0$. Just need to consider when $n=0$. Here $Hom(\mathbb{R},\mathbb{R}) = \mathbb{R}$. $H^0(pt;\mathbb{R}) = \frac{Ker(\delta)}{Im(\delta)}$ where $Im(\delta) = 0$ and $Ker(\delta) = \mathbb{R}$. Thus $H^0(pt;\mathbb{R}) = \mathbb{R}$. Hence we get our result. □

I now need to look at a specific type of exact sequence called a Mayer-Vietoris sequence which will be an important tool used in the proof of the De Rham theorem.

Definition 5.11 (Mayer-Vietoris sequence). Let U_1 and U_2 be open sets such that $U = U_1 \cup U_2$. Then the sequence

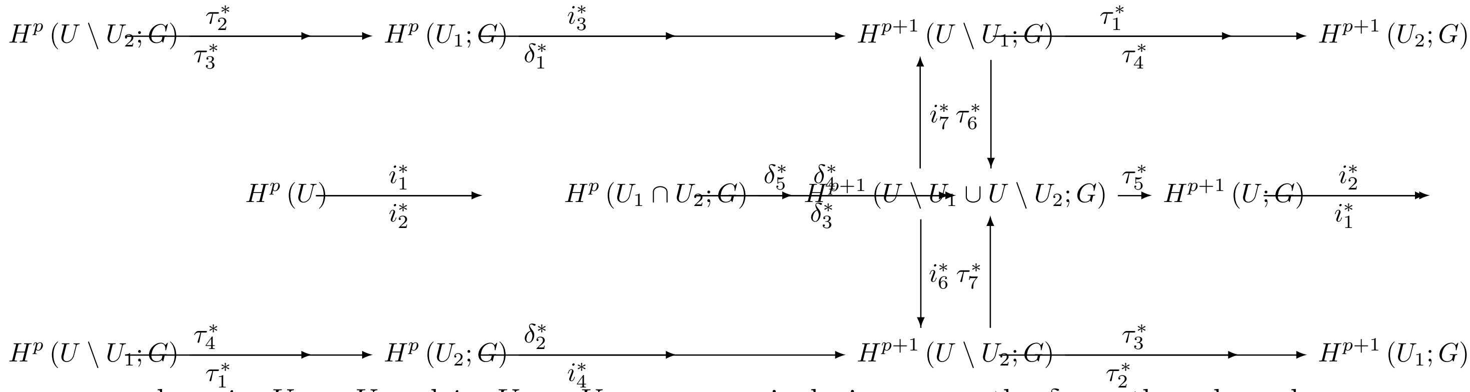
$$\dots \xrightarrow{\quad\delta\quad} H^p(U) \xrightarrow{\varphi} H^p(U_1) \oplus H^p(U_2) \xrightarrow{\psi} H^p(U_1 \cap U_2) \xrightarrow{\delta} H^{p+1}(U) \xrightarrow{\quad\varphi\quad} \dots$$

is called a Mayer-Vietoris sequence. φ and ψ will be defined in context.

I now want to prove a theorem which shows that Mayer-Vietoris sequences are exact.

Theorem 5.12. *Let U_1 and U_2 be open sets such that $U = U_1 \cup U_2$. Then the Mayer-Vietoris sequence of this decomposition is exact.*

Proof. We need the following diagram:



where $i_1 \colon U_1 \to U$ and $i_2 \colon U_2 \to U$ are reverse inclusion maps, the δ_i are the coboundary maps and the τ_i are homomorphisms.

We now define the following homomorphisms:

$$\varphi \colon H^p(U; G) \to H^p(U_1; G) \oplus H^p(U_2; G), \ \varphi(u) = (i_1^*(u), i_2^*(u))$$
$$\psi \colon H^p(U_1; G) \oplus H^p(U_2; G) \to H^p(U_1 \cap U_2), \ \psi(a, b) = i_3^*(a) - i_4^*(b)$$
$$\delta \colon H^p(U_1 \cap U_2; G) \to H^{p+1}(U; G), \ \delta(v) = \tau_1^* \delta_4^*(v)$$

These give rise to the following Mayer-Vietoris sequence:

$$\dots \xrightarrow{\quad\delta\quad} H^p(U; G) \xrightarrow{\varphi} H^p(U_1; G) \oplus H^p(U_2; G) \xrightarrow{\psi} H^p(U_1 \cap U_2; G) \xrightarrow{\delta} H^{p+1}(U; G) \xrightarrow{\quad\varphi\quad} \dots$$

It just remains to show that this is exact and this can be done by looking at the commutative diagram. To show it is an exact sequence we need to show the following:

(*i*) $Im(\varphi) = Ker(\psi)$

(*ii*) $Im(\varphi) = Ker(\delta)$

(*iii*) $Im(\delta) = Ker(\varphi)$

(*i*)$Im(\varphi) = H^p(U_1; G) \oplus H^p(U_2; G)$ since i_1 and i_3 are surjective. Now need to show that $Ker(\varphi) = H^p(U_1; G) \oplus H^p U_2; G$. An element (a, b) is mapped to 0 under ψ if $i_3(a) = i_4(b)$. This happens if the same element is mapped from $H^p(U; G)$ to each of $H^p(U_1; G)$ and $H^p(U_2; G)$. This is the case here and so $Im(\varphi) = Ker(\psi)$.
(*ii*)$Im(\psi) = H^p(U_1 \cap U_2)$ since i_3 and i_4 are surjective. Now need to show that $Ker(\delta) = H^p(U_1 \cap U_2 : G)$. So we need $\tau_1^* \delta_4^* = 0$. This is true since any element in $H^p(U_1 \cap U_2)$ will be mapped to a coboundary by the coboundary map δ_4^* and then to 0 by τ_1.
(*iii*) $Im(\delta) = 0$ and this is equal to $Ker(\varphi)$ since φ is surjective. ☐

6 Differential Forms on Manifolds

I am going to look at differential forms first before using them to define De Rham cohomology. They will be introduced in the context of integration on manifolds which will give some motivation for the De Rham theorem. The books used in this section were [4], [6], [7], [10], [11] and [13]. However we first need to cover tensors and alternating tensors before defining forms.

Definition 6.1 (A Tensor). Let V be a vector space over $\mathbb{R}$ of dimension k. A p-tensor T on V is a multi-linear map $T\colon V^p = \underbrace{V \times \ldots \times V}_{p-times} \to \mathbb{R}$. As T is multi-linear we have that

$$T\left(v_1, \ldots, v_j + av_j', \ldots, v_p\right) = T\left(v_1, \ldots, v_j, \ldots, v_p\right) + aT\left(v_1, \ldots, v_j', \ldots, v_p\right)$$

Here are some examples of tensors.

Example 6.2 (Examples of Tensors).
A 1-tensor is just a linear functional on V.
An example of a 2-tensor is the dot product on $\mathbb{R}^k$.
An example of a k-tensor is the determinant.

Here it is useful to define the space of all p-tensors and an exterior product on it.

Definition 6.3 (The space of all p-tensors). Let $T^p(V)$ be the set of all p-tensors on V.

Definition 6.4 (The tensor product). Let $T \in T^p(V)$ and $S \in T^q(V)$. Then the tensor product $\otimes\colon V^{p+q} \to \mathbb{R}$ is defined by

$$T \otimes S\left(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}\right) = T\left(v_1, \ldots, v_p\right) \cdot S\left(v_{p+1}, \ldots, v_{p+q}\right)$$

Now I would like to define an extra condition on a p-tensor.

Definition 6.5 (An alternating p-tensor). A p-tensor T is alternating if for all $\pi \in S_p$, $T^{\pi}\left(v_1, \ldots, v_p\right) = T\left(v_{\pi(1)}, \ldots, v_{\pi(p)}\right) = (-1)^{\pi} T$.

It is possible to turn any p-tensor T into an alternating tensor $Alt(T)$ by

$$Alt(T) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^{\pi} T^{\pi}$$

This is alternating since $(Alt(T))^{\sigma} = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^{\pi \circ \sigma} T^{\pi \circ \sigma} = (-1)^{\sigma} Alt(T)$ by setting $\tau = \pi \circ \sigma$ and noting that as π ranges through S_p so does τ. We also have the case that if T is already alternating then $Alt(T) = T$.

It now makes sense to define the space of all alternating p-tensors and an exterior product on this space.

Definition 6.6 (The space of alternating p-tensors). Let $Alt^p(V)$ be the vector space of all alternating p-tensors.

Definition 6.7 (The wedge product). Let $T \in Alt^p(V)$ and $S \in Alt^q(V)$. Now define the wedge product $\wedge\colon Alt^p(V) \times Alt^q(V) \to Alt^{p+q}(V)$ by $T \wedge S = Alt(T \otimes S)$.

I want to prove two important properties of the wedge product:

Lemma 6.8. *If $Alt(T) = 0$ then $T \wedge S = 0 = S \wedge T$.*

Proof. First note that S_{p+q} contains a subgroup G which fixes $p+1, \ldots, p+q$ and consists of all permutations of $1, \ldots, p$. We can assign to each $\pi \in G$ the permutation π' which is obtained by restricting π to $(1, \ldots, p)$. Also note that $(-1)^{\pi} = (-1)^{\pi'}$ and that $(T \otimes S)^{\pi} = T^{\pi'} \otimes S$. Hence

$$\sum_{\pi \in G} (-1)^{\pi} (T \otimes S)^{\pi} = \left[\sum_{\pi' \in S_p} (-1)^{\pi'} T^{\pi'} \right] \otimes S = Alt(T) \otimes S = 0$$

The subgroup G decomposes S_{p+q} into a disjoint union of right cosets $G \circ \sigma = \pi \circ \sigma\colon \pi \in G$. So for each coset we have:

$$\sum_{\pi \in G} (-1)^{\pi \circ \sigma} (T \otimes S)^{\pi \circ \sigma} = (-1)^{\sigma} \left[\sum_{\pi \in G} (-1)^{\pi} (T \otimes S)^{\pi} \right]^{\sigma} = 0$$

Since $T \wedge S = Alt(T \otimes S)$ is the sum of the partial summations over the cosets of G then $T \wedge S = 0$. The same proof can be mirrored for $S \wedge T = 0$. □

Theorem 6.9 (Associativity of the wedge product). *The wedge product is associative, ie*

$$(T \wedge S) \wedge R = T \wedge (S \wedge R)$$

Proof. Want to show that $(T \wedge S) \wedge R = Alt(T \oplus S \otimes R)$. By definition we have $(T \wedge S) \wedge R = Alt((T \wedge S) \otimes R)$ and as Alt is linear we have $(T \wedge S) \wedge R - Alt(T \otimes S \otimes R) = Alt([T \wedge S - T \otimes S] \otimes R)$. Since $T \wedge S$ is alternating we have $Alt(T \wedge S - T \otimes S) = Alt(T \wedge S) - Alt(T \wedge S) = T \wedge S - T \wedge S = 0$. So by lemma 6.8 we have $Alt([T \wedge S - T \otimes S] \otimes R) = 0$ as requires. The proof can be mirrored to give $T \wedge (S \wedge R) = Alt(T \otimes S \otimes R)$. □

I now want to show how alternating p-tensors can be studied on a manifold. This is done by defining a differential form on a manifold which is essentially a function which assigns to each point of the manifold an alternating p-tensor. A differential form is thus defined as follows:

Definition 6.10 (A differential form). Let M be a manifold then a p-form ω is a function which assigns, to every point $x \in M$, an alternating p-tensor $\omega(x) \in Alt^p(T_x M)$ where $T_x M$ is the tangent space of M.

Here the form is smooth if for any smooth vector fields $X_1, \ldots, X_p$ on M the function $\omega(X_1, \ldots, X_p)$ is smooth. It is also possible to define addition, multiplication and wedge products for forms. This is done pointwise:

$$(\omega_1 + \omega_2)(x) = (\omega_1)(x) + (\omega_2)(x)$$

$$(\lambda)(x) = \lambda(\omega)(x)$$

$$(\omega_1 \wedge \omega_2)(x) = (\omega_1)(x) \wedge (\omega_1)(x)$$

It is also desirable to define the space of all smooth p-forms.

Definition 6.11 (The space of smooth p-forms). The space of all smooth p-forms is a vector space and is denoted by $\Omega^p(M)$.

Here is an example of a 1-form.

Example 6.12. *Let $f\colon M \to \mathbb{R}$. Then df_x is a linear map $T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}$ is a 1-tensor and so the function $x \mapsto df_x$ is a 1-form.*

This is a very important example as it allows us to define forms in terms of the local coordinates on a manifold. Let (U, φ) be a chart with local coordinates $x_1, \ldots, x_n$. This gives 1-forms $dx_1, \ldots, dx_n$ as each x_i is a map $M \to \mathbb{R}$.

It is now possible to see what effect forms have on smooth maps. Let $f\colon X \to Y$ be a smooth map and ω be a smooth p-form on Y then it is possible to define a p-form $f^*\omega$ on X. If $f(x) = y$ then f induces a derivative map $df_x\colon T_x(X) \to T_y(Y)$. Now $\omega(y)$ is an alternating p-tensor on $T_y(Y)$ so it can be pulled back to $T_x(X)$ using the transpose $(df_x)^*$. The transpose map $A^*\colon W^* \to V^*$ is easily extended to give a map $A^*\colon Alt^p(W^*) \to Alt^p(V^*)\ \forall\, p \geq 0$. Now define $f^*\omega(x) = (df_x)^*\,\omega(f(x))$ so that $f^*\omega(x)$ is an alternating p-tensor on $T_x(X)$ and $f^*(x)$ is an p-form on X called the pullback of ω by f.

It is now possible to define the integration of forms.

Definition 6.13. Let $\omega = a dy_1 \wedge \ldots \wedge dy_n \in \Omega^n(V)$ where V is a smooth open subset of $\mathbb{R}^n$. Then $\int_V \omega = \int_V a dy_1 \wedge \ldots \wedge dy_n \in \mathbb{R}$.

Now using some of the results from section 2 it is possible to extend this definition on to a manifold.

Definition 6.14. Let M be an orientated manifold with a countable, locally finite, orientated atlas $\{(U_i, \varphi)\}$. Let $\{p_i\}$ be a partition of unity subordinate to $\{U_i\}$ where each $\overline{U_i}$ is compact. Given $\omega \in \Omega^n$ where $n = dim\ M$ define

$$\int_M \omega = \sum_{j=1}^{\infty} \int_{\varphi_j(U_j)} \left(\varphi_j^{-1}\right)^* (p_j\omega)$$

This is a locally finite sum

It just needs to be shown that this integral is independent of the choice of atlas and partition of unity.

Proposition 6.15. *This integral is independent of the choice of atlas and partition of unity.*

Proof. Let $\{(U_i, \varphi_1)\}$, $\{p_i\}$ and $\{(V_j, \phi_j)\}$, $\{\sigma_j\}$ be two choices of an atlas and a subordinate partition of unity. Then for each j,

$$\begin{aligned} \int_m \sigma_j\omega &= \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* (P_i\sigma_j\omega) \\ &= \sum_i \int_{\varphi_i(U_i \cap V_j)} \left(\varphi_i^{-1}\right)^* (p_i\sigma_j\omega) \end{aligned}$$

since $supp\,(p_i\sigma_j\omega) \subseteq U_i \cap V_j$. We have a diffeomorphism $\psi_j \circ \varphi_i^{-1} \colon \varphi_i (U_i \cap V_j) \to \psi_j (U_i \cap V_j)$ and that $det\left(d\left(\psi_j \circ \varphi_i^{-1}\right)\right) > 0$ by the orientation assumption. As this is greater than 0 we do not have to worry about putting any minus signs in. Thus

$$\begin{aligned}\int_M \sigma_j\omega &= \sum_i \int_{\psi_j(U_i\cap V_j)} \left(\varphi_i \circ \varphi_j^{-1}\right)^* \left(\varphi_i^{-1}\right) (p_i\sigma_j\omega) \\ &= \sum_i \int_{\psi_j(V_j)} \left(\psi_j^{-1}\right)^* (p_i\sigma_j\omega) \\ &= \int_{\psi_j(V_j)} \left(\psi_j^{-1}\right)^* (\sigma_j\omega)\end{aligned}$$

If the sum is taken over j both sides agree and we see that the integral is independent of the choice of atlas and partition of unity. □

I now want to look at an object called the exterior derivative which is essential to the proof of Stokes's theorem and will play a role analogous to the coboundary map in De Rham cohomology.

Definition 6.16 (Exterior derivative). Let $U \subset \mathbb{R}^n$ be open. Define an operator $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ and set $d\left(f dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. d is the exterior derivative.

It is now possible to prove Stokes's theorem. This is a very important theorem as from it all the classical theorems of Green, Gauß and Stokes follow along with the fundamental theorem of calculus. It also links analysis and topology as the boundary operator, ∂, is purely geometric while $\int$ and d are purely analytic.

Theorem 6.17 (Stokes's theorem). *Let M be an orientated manifold with boundary, dim $M = n$. Let $\omega \in \Omega^{n-1}(M)$. Then*

$$\int_{\partial M} \omega = \int_M d\omega$$

Proof. Both sides are linear in ω so we can assume that ω has compact support contained in some orientated chart (U, φ) for M, where $\varphi(U) \subset \mathbb{R}^n$ has compact closure. There are two cases:

Case 1 $\varphi(U) \subset Interior\,(H^n)$, ie $\varphi(U) \cap \partial H^n = \emptyset$

Then $\int_{\partial M} \omega = 0$ and $\int_M d\omega = \int_{\mathbb{R}^n} \left(\varphi^{-1}\right)^* (d\omega) = \int_{\mathbb{R}^n} d\left(\left(\varphi^{-1}\right)^*\right)$. We can write $\left(\varphi^{-1}\right)^* \omega = \sum_i (-1)^{i-1} a_i dx_i \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$. Here $\widehat{dx_i}$ means that dx_i is omitted from the product. Then $d\left(\left(\varphi^{-1}\right)^* \omega\right) = \sum_i \frac{\partial a_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$ and so

$$\begin{aligned}\int_M d\omega &= \int_{\mathbb{R}^n} \sum_i \frac{\partial a_i}{\partial x_i} dx_1 \ldots dx_n \\ &= \sum_i \int_{\mathbb{R}^n} \frac{\partial a_i}{\partial x_i} dx_1 \ldots\ dx_n\end{aligned}$$

The integral over $\mathbb{R}^n$ can be computed in the standard way as an iterated sequence of integrals over $\mathbb{R}^1$, which may be taken in any order by Fubini's theorem. So integrating the i^{th} term with respect to x_i:

$$\int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} dx_i \right) dx_1 \ldots \widehat{dx_i} \ldots dx_n$$

For $\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} dx_i$ treat the other x_i's as constants and note that $a_i = 0$ outside the compact set. This implies that $\int_{-R}^{R} \frac{\partial a_i}{\partial x_i} dx_i = 0$ for large R by the fundamental theorem of calculus. Thus $\int_M d\omega = 0$.

Case 2 $\varphi(U)$ meets ∂H^n

Write $(\varphi^{-1})^* \omega$ as above. This gives

$$\int_M d\omega = \sum_i \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} dx_i \right) dx_1 \ldots \widehat{dx_i} \ldots dx_n \text{ for } i < n$$

and this is zero as before. So only need to consider

$$\begin{aligned} \int_M d\omega &= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial a_n}{\partial x_n} dx_n \right) dx_1 \ldots dx_{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_0^{\infty} \frac{\partial a_n}{\partial x_n} \right) dx_1 \ldots dx_{n-1} \end{aligned}$$

Fixing $x_1, \ldots, x_{n-1}$ we get $a_n = 0$ for large x_n so the fundamental theorem of calculus gives us $\int_0^{\infty} \frac{\partial a_n}{\partial x_n} = -a_n(x_1, \ldots, x_{n-1}, 0)$. So

$$\int_M d\omega = \int_{\mathbb{R}^{n-1}} -a_n(x_1, \ldots, x_{n-1})\, dx_1 \ldots dx_{n-1}$$

Now consider the inclusion map $i\colon \mathbb{R}^{n-1} \to \partial H^n \subset \mathbb{R}^n,\ x \mapsto (x, 0)$. Then $i^*(dx_n) = d(i^* x_n) = d\left(n^{\text{th}} \text{ coordinate of } i(x)\right) = 0$. We also have that $(\varphi^{-1})^* \omega = \sum_i (-1)^{i-1} a_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$ and $i^*(\varphi^{-1*}\omega) = (-1)\, i^* a_n dx_1 \wedge \ldots \wedge dx_{n-1}$. Hence

$$\begin{aligned} \int_{\partial M} \omega &= \int_{\partial H^n} (-1)^{n-1} a_n(x_1, \ldots, x_{n-1}, 0)\, dx_1 \wedge \ldots \wedge dx_{n-1} \\ &= (-1)^{n-1} (-1)^n \int_{\mathbb{R}^n} a_n(x_1, \ldots, x_{n-1}, 0)\, dx_1 \ldots dx_{n-1} \\ &\quad \text{(extra } (-1)^n \text{ comes from the fact that } i \text{ changes orientation by that factor)} \\ &= \int_M \partial\omega \end{aligned}$$

□

With Stokes's theorem I have all the necessary tools from calculus to start looking at De Rham cohomology.

7 De Rham Cohomology

In this section I am going to build up the theory of De Rham cohomology. The main tool used in De Rham cohomology are the differential forms looked at in the previous section. The theory of De Rham cohomology is going to be very similar to singular cohomology and this will provide some of the motivation for the De Rham theorem. The books used in this section were [4], [5], [7], [8], [9] and [13]. I want to start by defining the De Rham cohomology group.

Definition 7.1 (The p^{th} De Rham cohomology group). The p^{th} De Rham cohomology group of U is the quotient space

$$H^p(U) = \frac{Ker\left(d \colon \Omega^p(U) \to \Omega^{p+1}(U)\right)}{IM\left(d \colon \Omega^{p-1}(U) \to \Omega^p(U)\right)}$$

In particular $H^p(U) = 0$ for $p < 0$ and $H^0(U) = Ker\left(d \colon C^\infty(U;\mathbb{R}) \to \Omega^1(U)\right)$.

For this definition to make sense we need $d \circ d = 0$. This is true since from our definition of d we have $d\left(f dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and when we apply $d \circ d$ to $f dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ we get $d\left(df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right) = 0$ since there is no function for d to act on, so it maps the entire thing to 0.

I now want to show how a smooth map can induce a morphism on the space of alternating p-forms. This is done in the following way:

Definition 7.2. Let $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^n$ be open sets and $\phi \colon U_1 \to U_2$ a smooth map. The induced morphism $\Omega^p(\phi) : \Omega^p(U_2) \to \Omega^p(U_1)$ is defined by

$$\Omega^p(\phi)(\omega)(x) = Alt^p(D_x\phi) \circ \omega(\phi(x)) \text{ where } \Omega^0(\phi)(\omega)(x) = \omega(\phi(x))$$

I now want to look at a specific example of the morphism induced by a smooth map.

Example 7.3. *Let $\phi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is given by $\phi(x,t) = \phi(t)\,x$, where $\phi(t)$ is a smooth real valued function. Then*

$$\Omega^p(\phi)(dx_i) = x_i\phi'(t)\,dt + \phi(t)\,dx_i$$

Now given a smooth map $\phi \colon U_1 \to U_2$ it is possible to associate a linear map

$$H^p(\phi) : H^p(U_2) \to H^P(U_1)$$

by setting $H^p(\phi)[\omega] = [\Omega^p(\phi)(\omega)]$. This is independent of the choice of representative, since $\Omega^p(\phi)(\omega + dv) = \Omega^p(\phi)(\omega) + \Omega^p(\phi)(dv) = \Omega^p(\phi)(\omega) + d\Omega^p(\phi)(v)$. Also,

$$H^{p+q}(\phi)([\omega_1][\omega_2]) = (H^p(\phi)[\omega_1])(H^q(\phi)[\omega_2])$$

Now I want to prove a very famous lemma called the Poincaré lemma. It will be useful for calculation of De Rham cohomology groups.

Lemma 7.4 (Poincaré's lemma). *If U is a star-shaped open set then $H^p(U) = 0$ for $p > 0$ and $H^0(U) = \mathbb{R}$.*

Proof. Firstly, assume U is a star shaped with respect to the origin $0 \in \mathbb{R}^n$. Now want to construct a linear operator $S_p \colon \Omega^p(U) \to \Omega^{p-1}(U)$ such that the following holds:

$$dS_p + S_{p+1}d = id \text{ when } p > 0$$

and

$$S_1 d = id - e$$

where $e(\omega) = \omega(0)$ for $\omega \in \Omega^0 U$. If such an operator exists then the lemma is true because $dS_p(\omega) = \omega$ for a closed p-form, $p > 0$ and thus $[\omega] = 0$. If $p = 0$ we get $\omega - \omega(0) = S_1 d\omega = 0$ and ω must be a constant. The first stage in constructing our linear operator is to define an operator $\widehat{S}_p \colon \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$. Now for every $\omega \in \Omega(U \times \mathbb{R})$ it can be written in the form $\omega = \sum f_I(x,t)\, dx_I + \sum g_J(x,t)\, dx_t \wedge dx_J$ where $I = (i_1, \ldots i_p)$ and $J = (j_1, \ldots, j_p)$. Now define $\widehat{S}_p(\omega) = \sum \left(\int_0^1 g_J(x,t)\, dt \right) dx_J$. We now have

$$\begin{aligned}
d\widehat{S}_p(\omega) + \widehat{S}_{p+1}d(\omega) &= \sum_{J,i} \left(\int_0^1 \frac{\partial g_J(x,t)}{\partial x_i} \right) dx_i \wedge dx_J \\
&= \sum_I \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t} dt \right) dx_I \\
&- \sum_{J,i} \left(\int_0^1 \frac{\partial g_J}{\partial x_i} dt \right) dx_i \wedge dx_J \\
&= \sum \left(\int_0^1 \frac{\partial f_I(x,t)}{\partial t} \right) dx_I \\
&= \sum f_i(x,1)\, dx_I - \sum f_I(x,0)\, dx_I
\end{aligned}$$

Let $\phi \colon U \times \mathbb{R} \to U,\ \phi(x,t) = \psi(t)\, x$ where $\phi(t) = \begin{cases} \phi(t) = 0 & \text{if } t \leq 0 \\ \phi(t) = 1 & \text{if } t \geq 1 \\ 0 \leq \phi(t) \leq 1 & \text{otherwise} \end{cases}$. Now consider the action of $S_p(\omega)$ on $\Omega^p(\phi)$, we define this in the following way $S_p(\omega) = \widehat{S}_p(\Omega^p(\phi)(\omega))$ where $\widehat{S}_p \colon \Omega^p(U \times \mathbb{R}) \to \Omega^{p-1}(U)$ as above. Now assume that $\omega = \sum h_I(x)\, dx_I$ and from example 7.3 we get

$$\Omega^p(\phi)(\omega) = \sum h_I(\psi(t)\, x)\, (d\psi(t)\, s_{i_1} + \psi(t)\, dx_{i_1}) \wedge \ldots \wedge \left(d\psi(t)\, x_{i_p} + \psi(t)\, dx_{i_p} \right)$$

So now we have, using the same notation as above

$$\sum f_I(x,t)\, dx_I = \sum h_I(\psi(t)\, x)\, \psi(t)^p\, dx_I$$

Thus we have $dS_p(\omega) + S_{p+1}d(\omega) = \begin{cases} \sum_I h_I(x)\, dx_I = \omega & p > 0 \\ \omega(x) - \omega(0) & p = 0 \end{cases}$. □

Now I need to calculate one specific example which will be needed in the proof of the De Rham theorem.

Example 7.5 (The De Rham cohomology of $\mathbb{R}^n$). *We know that $\mathbb{R}^n$ is a star-shaped open set and so we get* $H^p(\mathbb{R}^n) = \begin{cases} 0 & p > 0 \\ \mathbb{R} & p = 0 \end{cases}$.

In section 5 I showed how to derive the Mayer-Vietoris sequence for singular cohomology. I would now like to show how the analogous sequence for De Rham cohomology is set up. Firstly I want to prove the following result about sequences of alternating groups.

Theorem 7.6. *Let U_1 and U_2 be open sets of $\mathbb{R}^n$ with union $U = U_1 \cup U_2$. For $\nu = 1, 2$, let $i_\nu \colon U_\nu \to U$ and $j_\nu \colon U_1 \cap U_2 \to U_\nu$ be the corresponding inclusions. Then the sequence*

$$0 \longrightarrow \Omega^p(U)\, 8 \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega(U_1 \cap U_2) \longrightarrow 0$$

is exact, where $I^p(\omega) = (\Omega^p(i_1)(\omega), \Omega^p(i_2)(\omega))$, $J^p(\omega_1, \omega_2) = \Omega^p(j_1)(\omega_1) - \Omega^p(j_2)(\omega_2)$.

Proof. For a smooth map $\phi \colon V \to W$ and a p-form $\omega = \sum f_I dx_I \in \Omega^p(W)$,

$$\Omega^p(\phi)(\omega) = \sum (f_I \circ \phi)\, d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p}$$

So, if ϕ is an inclusion of open sets in $\mathbb{R}^n$, ie $\phi_i(x) = x_i$, then

$$d\phi_{i_1} \wedge \ldots \wedge d\phi_{i_p} = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

Hence

$$\Omega^p(\phi)(\omega) = \sum (f_I \circ \phi)\, dx_I \tag{3}$$

Then consider $\phi = i_\nu, j_\nu$ for $\nu = 1, 2$. It follows from (3) that I^p is injective, ie $I^p(\omega) = 0$ then $\Omega^p(i_\nu)(\omega) = \sum (f_I \circ \phi)\, dx_I = 0$ if and only if $f_I \circ i_\nu = 0$ for all I. However $f_I \circ i_1 = 0$ and $f_I \circ i_2 = 0$ gives us that $f_I = 0$ on the whole of U, since U_1 and U_2 form a cover of U. We can show that $Ker\left(J^P\right) = Im\,(I^p)$. Firstly note that

$$J^p \circ I^p(\omega) = \Omega^p(j_2)\,\Omega^p(i_2)(\omega) - \Omega^p(j_1)\,\Omega^p(i_1)(\omega) = \Omega^p(j)(\omega) - \Omega^p(j)(\omega) = 0$$

where $j \colon U_1 \cap U_2 \to U$ is an inclusion. So we have $Im\,(I^p) \subseteq Ker\,(J^p)$. Now we have to show the inverse inclusion. So given two p-forms $\omega_\nu \in \Omega^p(U_\nu)$, where $\omega_1 = \sum f_I dx_I$ and $\omega_2 = \sum g_I dx_I$. We know that $J^p(\omega_1, \omega_2) = 0$ and $\Omega^p(j_1)(\omega_1) = \Omega^p(j_2)(\omega_2)$, which by (3) gives us that $f_I \circ j_1 = g_I \circ j_2$ or $f_I(x) = g_I(x)$ for $x \in U_1 \cap U_2$. We thus define a smooth function $h_I \colon U \to \mathbb{R}^n$ by

$$h_I(x) = \begin{cases} f_I(x) & x \in U_1 \\ g_I(x) & x \in U_2 \end{cases}$$

Then $I^p\left(\sum h_I dx_I\right) = (\omega_1, \omega_2)$. It remains to show that J^p is surjective. By section 2, we can obtain a partition of unity $\{p_1, p_2\}$ with support in $\{U_1, U_2\}$. Let $f \colon U_1 \cap U_2 \to \mathbb{R}$ be a smooth function. Now use $\{p_1, p_2\}$ to extend f to U_1 and U_2 in the following way. Since $supp_U(p_1) \cap U_2 \subset U_1 \cap U_2$ it is possible to define a smooth function

$$f_2(x) = \begin{cases} -f(x)\, p_1(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_2 \setminus supp_U(p_1) \end{cases}$$

Analogously we have

$$f_1(x) = \begin{cases} f(x)\,p_2(x) & \text{if } x \in U_1 \cap U_2 \\ 0 & \text{if } x \in U_1 \setminus supp_U(p_2) \end{cases}$$

Now $f_1(x) - f_2(x) = f(x)$ when $x \in U_1 \cap U_2$ since $P_1(x) + p_2(x) = 1$. Given a differential form $\omega \in \Omega^p(U_1 \cap U_2)$, $\omega = \sum f_I dx_I$, the above argument can be applied to each $f_I \colon U_1 \cap U_2 \to \mathbb{R}$. This gives functions $f_{I,\nu} \colon U_\nu \to \mathbb{R}$ and differential forms $\omega_\nu = \sum f_{I,\nu} dx_I \in \Omega(U_\nu)$. This gives $J^p(\omega_1, \omega_2) = \omega$. □

I need one more lemma before I can prove the existence of Mayer-Vietoris sequences for De Rham cohomology.

Lemma 7.7. *If A^* and B^* are cochain complexes then $H^p(A^* \oplus B^*) = H^P(A^*) \oplus H^p(B^*)$.*

Proof. Here we have that $Ker\left(d^p_{A\oplus B}\right) = Ker\left(d^p_A\right) \oplus Ker\left(d^p_B\right)$ and $Im\left(d^{p-1}_{A\oplus B}\right) = Im\left(d^{p-1}_A\right) \oplus Im\left(d^{p-1}_B\right)$. The lemma follows. □

Here is the theorem which shows the existence of Mayer-Vietoris sequences for De Rham cohomology.

Theorem 7.8 (Existence of Mayer-Vietoris sequences for De Rham cohomology). *Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and $U = U_1 \cup U_2$. Then there exists an exact sequence of cohomology vector spaces*

$$\cdots \longrightarrow H^p(U) \xrightarrow{\Omega^p(I)} H^p(U_1) \oplus H^p(U_2) \xrightarrow{\Omega^p(J)} H^p(U_1 \cap U_2) \xrightarrow{\Omega^p(\partial)} H^{p+1}(U) \longrightarrow \cdots$$

where $\Omega^p(I)([\omega]) = (\Omega^p(i_1)[\omega], \Omega^p(i_2)[\omega])$ and $\Omega^p(J)([\omega_1],[\omega_2]) = \Omega^p(j_1)[\omega_1] - \Omega^p(j_2)[\omega_2]$ in the notation of theorem 7.6.

Proof. Apply theorem 7.6 to the maps of cochain complexes

$$\begin{aligned} I &: \Omega^p(U) \to \Omega^p(U_1) \oplus \Omega^p(U_2) \\ J &: \Omega^p(U_1) \oplus \Omega^p(U_2) \to \Omega^p(U_1 \cap U_2) \end{aligned}$$

to obtain a short exact sequence of De Rham cohomology vector spaces. Now apply lemma 7.7, which says that

$$H^p(\Omega^p(U_1) \oplus \Omega^p(U_2)) = H^p(\Omega^p(U_1)) \oplus H^p(\Omega^p(U_2))$$

The theorem follows. □

I now want to move onto the main part of the project and prove the De Rham theorem.

8 The De Rham Theorem

The books used in this section were [5], [7], [8], [9], [12] and [13]. The following journal articles were also consulated [1] and [2]. I want to start off by stating the De Rham theorem before explaining what it means and its implications.

Theorem 8.1 (The De Rham theorem). *For any manifold M, the cochain map φ induces a natural isomorphism $\varphi^*\colon H^p(M) \to H^n(M;\mathbb{R})$ of cohomology groups.*

I will define the map φ in a bit but firstly I want to comment on the De Rham theorem and some of it's implications.

The De Rham theorem answers one of the first questions which springs to mind when discussing singular and De Rham cohomology. Are they related in any way? The De Rham theorem says that they will produce the same group to within an isomorphism. In fact all of the cohomology theories, of which singular and De Rham are but examples produce the same group to within an isomorphism. This is quite an amazing fact as singular cohomology is a very topological theory as was shown in section 5, it is derived from looking at simplicial complexes and cochains on them. De Rham cohomology is much more analytical. It comes from differential forms on manifolds and the exterior derivative. It is perhaps strange to see that analysis and topology can be linked in such a nice way, however Stokes's theorem in chapter 6 also shows this relationship with the left hand side of the equation coming from topology and the right hand side coming from analysis.

Firstly I want to define how to integrate over a simplex.

Definition 8.2. Let $T\colon \sigma \to X$ be a singular p-simplex in X and ω a differential form. We can define the integral of ω over T, $\int_T \omega$, as follows. Firstly note that $\Omega^p(T)(\omega) = f dx_1 dx_2 \dots dx_n$. We can now take $\int_T \omega$ to be the n-fold integral of the C^∞ real valued function f over the simplex σ

I now want to define the map φ and also explain some of the consequences of it being an isomorphism.

Definition 8.3. Let $\varphi\colon \Omega^n(M) \to Hom(Q_p(M), \mathbb{R})$ be a homomorphism defined by $\langle \varphi\omega, u\rangle = \int_u \omega$ where $Q_p(M)$ is the free Abelian group generated by the set of all n-simplexes in M and $u = \sum a_i T_i$.

Here we have $\int_u \omega = \sum a_i \int_{T_i} \omega$.

By theorem 6.17, we have that φ is in fact a cochain map

$$\Omega^p(M) \to Hom(Q_p(M); \mathbb{R})$$

This map φ allows us to have an analytic interpretation of the De Rham theorem. The De Rham theorem is saying that we can take any form on any manifold and it is possible to integrate it by mapping it onto a simplex and integrating it there and the result will be the same as if we had integrated it on the manifold. This is a very useful result as integrals on non-orientated manifolds are not defined but integrals on simplexes are.

I now want to prove the De Rham theorem.

Proof. I want to use induction on open sets to prove the De Rham theorem. This requires use of the main result of section 3; theorem 3.1. There are 4 steps that have to be shown to be able to prove the De Rham theorem.

Step 1 $\emptyset$ satisfies the De Rham theorem

Step 2 Any open subset $U \subseteq V_\beta$ diffeomorphic to $\mathbb{R}^n$ satisfies the De Rham theorem

Step 3 If $U_1, U_2, U_1 \cap U_2$ satisfy the De Rham theorem then so does $U_1 \cup U_2$

Step 4 If $U_1, U_2, \ldots$ is a sequence of pairwise disjoint subsets with each U_i satisfying the De Rham theorem then so does their union $\bigcup_i U_i$

The result then follows from theorem 3.1.

Step 1 Here $H^p(\emptyset) = 0$ as there are no p-forms on the empty set and $H^p(\emptyset; \mathbb{R}) = 0$ as there are no homomorphisms from the empty set to the reals.

Step 2 Firstly note that diffeomorphisms preserve the cohomology structure so we have $H^p(U; \mathbb{R}) \cong H^p(\mathbb{R}^n; \mathbb{R})$ and $H^p(U) \cong H^p(\mathbb{R}^n)$. Proposition 5.10 and example 7.5 tell us that $H^p(\mathbb{R}^n; \mathbb{R}) = \begin{cases} 0 & \text{if } p > 0 \\ \mathbb{R} & \text{if } p = 0 \end{cases}$ and $H^p(\mathbb{R}^n) = \begin{cases} 0 & \text{if } p > 0 \\ \mathbb{R} & \text{if } p = 0 \end{cases}$ Now need to show that φ is an isomorphism. In the case $p = 0$ we are only integrating a 0-form which is a continuous function, f, so our integral is the value at a point which is the value of f integrated at that point. This is not the zero map and so φ is an isomorphism. When $p > 0$, φ clearly maps 0 to 0 and so is an isomorphism.

Step 3 Here the proof requires the setting up of two Mayer-Vietoris sequences and showing that φ links them

Let $i\colon U_1 \cap U_2 \to U_1$, $j\colon U_1 \cap U_2 \to U_2$, $k\colon U_1 \to M$ and $l\colon U_2 \to M$ be inclusion maps.

Define cochain maps $\alpha \Omega^p(U_1 \cup U_2) \to \Omega^p(U_1) \oplus \Omega^p(U_2)$ and $\beta\colon \Omega^p(U_1) \oplus \Omega^p(U_2)$ by $\alpha(\omega) = (\Omega^p(k)(\omega), \Omega^p(l)(\omega))$ and $\beta(\omega_1, \omega_2) = \Omega^p(i)(\omega_1) - \Omega^p(j)(\omega_2)$.

Now we need to show that the following sequence is exact:

$$\ldots \longrightarrow \Omega^p(U_1 \cup U_2) \xrightarrow{\alpha} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{\beta} \Omega^p(U_1 \cap U_2) \longrightarrow \ldots$$

So we need to show that α is injective and β is surjective.

It is clear that α is injective. So it only remains to show that β is surjective. This is done as follows:

Let $\{g, h\}$ be a C^∞ partition of unity subordinate to the open covering $\{U_1, U_2\}$ of M. Let ω be a differential form on $U_1 \cap U_2$. Then $g\omega$ can be extended to a C^∞ differential form ω_{U_2} on U_2 be defining $\omega_{U_2}(x) = 0$ at any $x \in U_2 \setminus U_1$. Similarly $h\omega$ can be extended to a C^∞ differential form on U_1 by defining $\omega_{U_1}(y) = 0$ at any point $y \in U_1 \setminus U_2$. Then $\beta(\omega_{U_1} - \omega_{U_2}) = 0$.

This sequence then gives rise to the following Mayer-Vietoris sequence.

$$\ldots \longrightarrow H^p(U_1 \cup U_2) \xrightarrow{\alpha} H^p(U_1) \oplus H^p(U_2) \xrightarrow{\beta} H^p(U_1 \cap U_2) \xrightarrow{d} H^{p+1}(U_1 \cup U_2) \longrightarrow \ldots$$

The same thing is done for singular cohomology. Here the required short sequence is:

$$\ldots \longrightarrow H^p(U_1 \cup U_2; \mathbb{R}) \xrightarrow{\alpha'} Hom(U_1; \mathbb{R}) \oplus Hom(U_2; \mathbb{R}) \xrightarrow{\beta'} Hom(U_1 \cap U_2; \mathbb{R}) \longrightarrow \ldots$$

Here $\alpha'(f) = (fk, fl)$ and $\beta'(f_1, f_2) = f_1 i - f_2 j$. Still need to show α' is injective and β' is surjective. It is clear that this is the case.

This exact sequence gives rise to the following Mayer-Vietoris sequence:

$$\ldots \longrightarrow H^p\left(U_1 \cup U_2; \mathbb{R}\right) \xrightarrow{\alpha'} H^p\left(U_1; \mathbb{R}\right) \oplus H^p\left(U_2; \mathbb{R}\right) \xrightarrow{\beta'} H^p\left(U_1 \cap U_2; \mathbb{R}\right) \xrightarrow{\partial} H^{p+1}\left(U_1 \cup U_2; \mathbb{R}\right) \longrightarrow \ldots$$

These diagrams can be linked by the homomorphism φ in the following way:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & H^p\left(U_1 \cup U_2\right) & \xrightarrow{\alpha} & H^p\left(U_1\right) \oplus H^p\left(U_2\right) & \xrightarrow{\beta} & H^p\left(U_1 \cap U_2\right) & \xrightarrow{d} & H^{p+1}\left(U_1 \cup U_2\right) & \longrightarrow & \ldots \\
 & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\
\ldots & \longrightarrow & H^p\left(U_1 \cup U_2; \mathbb{R}\right) & \xrightarrow{\alpha'} & H^p\left(U_1; \mathbb{R}\right) \oplus H^p\left(U_2; \mathbb{R}\right) & \xrightarrow{\beta'} & H^p\left(U_1 \cap U_2; \mathbb{R}\right) & \xrightarrow{\partial} & H^{p+1}\left(U_1 \cup U_2; \mathbb{R}\right) & \longrightarrow & \ldots
\end{array}$$

Firstly need to show that the diagram is commutative.

Given a p-form ω this is mapped to α to $\left(\Omega^p\left(k\right)\left(\omega\right), \Omega^p\left(l\right)\left(\omega\right)\right)$. This is then mapped by φ to $\left(\int_T \Omega^p\left(k\right)\left(\omega\right), \int_T \Omega^p\left(l\right)\left(\omega\right)\right)$. Going the other way around the square, we have ω mapped to $\int_T \omega$ which is then mapped to $\left(\int_T k\left(\omega\right), \int_T l\left(\omega\right)\right)$ under α'. These are identical as we are mapping functions under α' and forms under α.

Consider an element of $H^p\left(U_1\right) \oplus H^p\left(U_2\right),\ \left(\omega_1, \omega_2\right)$. This is mapped to $\Omega^p\left(I\right)\left(\omega_1\right) - \Omega^p\left(j\right)\left(\omega_2\right)$ under β. This is mapped to $\int_T \left(\Omega^p\left(i\right)\left(\omega_1\right) - \Omega^p\left(j\right)\left(\omega_2\right)\right)$ under φ. Going the other way round the square, φ maps $\left(\omega_1, \omega_2\right)$ to $\left(\int_T \omega_1, \int_T \omega_2\right)$ which is then mapped to $\int_T \left(i\left(\omega_1\right) - j\left(\omega_2\right)\right)$ by β'. Again these are the same as the map β does the same thing to forms as β' does to functions.

Consider an element of $H^p\left(U_1 \cap U_2\right),\ \omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. This is mapped to $\omega' = df \wedge dx_{i_1} \wedge \ldots dx_{i_p}$. By φ this is mapped to $\int_T df\, dx_{i_1} \ldots dx_{i_p}$. Now consider going the other way around the square. Firstly, φ maps ω to $\int_T \omega = \int_T f dx_{i_1} \ldots dx_{i_p}$. This is then mapped to $\int_T df\, dx_{i_p} \ldots dx_{i_p}$.

Thus the diagram is commutative.

By assumptions only need to show that $\varphi\colon H^p\left(U_1 \cup U_2\right) \to H^p\left(U_1 \cup U_2; \mathbb{R}\right)$ is an isomorphism. This follows straight from theorem 4.2 as the two maps on both sides of $\varphi\colon H^p\left(U_1 \cup U_2\right) \to H^p\left(U_1 \cup U_2\right)$ are isomorphisms by assumption.

This can be extended to an induction argument to hold for all p, however the case $p = 1$ needs to be proved separately. Here we need the following diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H^1\left(U_1 \cup U_2\right) & \xrightarrow{\alpha} & H^1\left(U_1\right) \oplus H^1 bracm U_2 & \xrightarrow{\beta} & H^1\left(U_1 \cap U_2\right) & \longrightarrow & \ldots \\
\downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi & & \\
0 & \longrightarrow & 0 & \longrightarrow & H^1\left(U_1 \cup U_2; \mathbb{R}\right) & \xrightarrow{\alpha'} & H^1\left(U_1; \mathbb{R}\right) \oplus H^1\left(U_2; \mathbb{R}\right) & \xrightarrow{\beta'} & H^1\left(U_1 \cap U_2; \mathbb{R}\right) & \longrightarrow & \ldots
\end{array}$$

To be able to apply the five lemma (theorem 4.2) here, need to show that $\varphi\colon 0 \to 0$ is an isomorphism. φ maps 0 to 0 and so is an isomorphism. Thus the five lemma applies and so $\varphi\colon H^1\left(U_1 \cup U_2\right) \to H^1\left(U_1 \cup U_2; \mathbb{R}\right)$ is an isomorphism.

Thus $\varphi\colon H^p\left(U_1 \cup U_2\right) \to H^p\left(U_1 \cup U_2; \mathbb{R}\right)$ is an isomorphism for all p and so step 3 is proved.

Step 4 Here we need to show that the following diagram commutes isomorphically:

$$\begin{array}{ccc} H^p(U) & \xrightarrow{\gamma} & \prod_{\alpha\in A} H^p(U_\alpha) \\ \varphi \downarrow & & \downarrow \prod_{\alpha\in A}\varphi_\alpha \\ H^p(U;\mathbb{R}) & \xrightarrow{\gamma'} & \prod_{\alpha\in A} H^p(U_\alpha;\mathbb{R}) \end{array}$$

Where $\gamma\colon H^p(U) \to \prod_{\alpha\in A} H^p(U_\alpha)\,,\ [\omega] \mapsto [\Omega^p(i_\alpha)(\omega)]\,,\ i_alpha\colon U_\alpha \to U$ is an inclusion map and $\gamma'\colon H^p(U;\mathbb{R}) \to \prod_{\alpha\in A} H^p(U_\alpha;\mathbb{R})\,,\ f \mapsto fi_\alpha$. Also φ_α is just the usual map φ but going from H^pU_α to $H^p(U_\alpha;\mathbb{R})$.

Need to show both γ and γ' are both isomorphisms. For γ consider each $\Omega^p(i_\alpha)$ unto the whole of $\prod_{\alpha\in A} H^p(U_\alpha)$ by defining it as follows:

$$\Omega^p(i_\alpha)(\omega) = \begin{cases} \Omega^p(i_\alpha)(\omega) & \text{on } H^p(U_\alpha) \\ 0 & \text{on } \prod_{\beta\in A,\ \beta\neq\alpha} H^p(U_\beta) \end{cases}$$

When these are combined to give γ it will be an isomorphism.

For γ' consider each fi_α separately. Each of these is an isomorphism onto $H^p(U_\alpha;\mathbb{R})$ for each α. It is possible to extend each of these onto the whole of $\prod_{\alpha\in A} H^p(U_\alpha;\mathbb{R})$ by defining it as follows:

$$fi_\alpha = \begin{cases} fi_\alpha & \text{on } H^p(U_\alpha) \\ 0 & \text{on } \prod_{\beta\in A,\ \beta\neq\alpha} H^p(U_\beta;\mathbb{R}) \end{cases}$$

Thus have shown that each of the maps is an isomorphism apart from $\prod_{\alpha\in A}\varphi_\alpha$. This map is also an isomorphism as the diagram is commutative. This proves step 4 and thus the De Rham theorem. □

References

[1] M.L. Curtis and J. Dugundji. *A Proof of the De Rham Theorem.* Fundamenta Mathematica, pages 265-268. 1970

[2] V.K.A.M. Gugenheim. *On the multiplicative structure of the De Rham theory.* Journal of Differential Geometry, pages 309-314. 1976

[3] A.M. Armstrong. *Basic Topology.* Springer. 1983

[4] V. Guillemin and A. Pollack. *Differential Topology.* Prentince-Hall, Inc. 1974

[5] A. Hatcher. *Algebraic Topology.* Cambridge University Press. 2002

[6] S. Lang *Introduction to Differentiable Manifolds.* Interscience Publications. 1962

[7] I. Madsen and J. Tornehave. *From Calculus to Cohomology : De Rham Cohomology and Characteristic Classes.* Cambridge University Press. 1997

[8] W.S. Massey. *Homology and Cohomology Theory : An Approach Based on Alexander-Spanier Cochains.* Marcel Dekker, Inc. 1978

[9] W.S. Massey. *A Basic Course in Algebraic Topology.* Springer-Verlag. 1991

[10] M. Spivak. *Calculus on Manifolds : A Modern Approach to Classical Theorems of Advanced Calculus.* W.A. Benjamin, Inc. 1965

[11] M. Spivak. *A Comprehensive Introduction to Differential Geometry, Vol I.* Publish or Perish, Inc. 1970

[12] J.W. Vick. *Homology Theory : An Introduction to Algebraic Topology.* 1973

[13] F.W. Warner. *Foundations of Differential Manifolds and Lie Groups.* Springer-Verlag. 1971