On the Multimodal Logic of Normative Systems

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Abstract. We introduce Multimodal Logics of Normative Systems as a contribution to the development of a general logical framework for reasoning about normative systems over logics for Multi-Agent Systems. Given a multimodal logic L, for every modality $\Box_i$ and normative system η, we expand the language adding a new modality $\Box_i^\eta$ with the intended meaning of $\Box_i^\eta \phi$ being "ϕ is obligatory in the context of the normative system η over the logic L". In this expanded language we define the Multimodal Logic of Normative Systems over L, for any given set of normative systems N, and we give a sound and complete axiomatisation for this logic, proving transfer and model checking results. The special case when L and N are axiomatised by sets of Sahlqvist or shallow modal formulas is studied.

Keywords: Fusions of Logics, Multimodal Logics, Normative Systems, Multi-Agent Systems, Model Theory, Sahlqvist Formulas

1 Introduction

Recent research on the logical foundations of Multi-Agent Systems (MAS) has centered its attention in the study of normative systems. The notion of electronic institution is a natural extension of human institutions by permitting not only humans but also autonomous agents to interact with one another. Institutions are used to regulate interactions where participants establish commitments and to facilitate that these commitments are upheld, the institutional conventions are devised so that those commitments can be established and fulfilled (see [1] for a general reference of the role of electronic institutions to regulate agents interactions in MAS). Over the past decade, normative systems have been promoted for the coordination of MAS and the engineering of societies of self-interested autonomous software agents. In this context there is an increasing need to find a general logical framework for the study of normative systems over the logics for MAS.

Given a set of states S and a binary accessibility relation R on S, a normative system η on the structure (S, R) could be understood as a set of constraints $\eta \subseteq R$ on the transitions between states, the intended meaning of $(x, y) \in \eta$ being "the transition from state x to state y is not legal according to normative system η". Several formalisms have been introduced for reasoning about normative systems over specific logics, two examples are worth noting: Normative ATL (NATL), proposed in [2] and Temporal

Logic of Normative Systems (NTL) in [3]. NATL is an extension to the Alternating-Time Temporal Logic and contains cooperation modalities of the form $<< \eta : C >> \phi$ with the intended interpretation that "C has the ability to achieve ϕ within the context of the normative system η". NTL is a conservative generalization of the Branching-Time Temporal Logic CTL. In NTL, the path quantifiers A ("on all paths...") and E ("on some path...") are replaced by the indexed deontic operators O_η ("it is obligatory in the context of the normative system η that..") and P_η ("it is permissible in the context of the normative system η that...").

The Multimodal Logic of Normative Systems introduced in this article is a contribution to define a general logical framework for reasoning about normative systems over logics for MAS, for this purpose we generalize to arbitrary logics the approaches taken in [2] and [3]. At the moment, we are far from obtaining a unique formalism which addresses all the features of MAS at the same time, but the emerging field of combining logics is a very active area and has proved to be successful in obtaining formalisms which combine good properties of the existing logics. In our approach, we regard the Logic of Normative Systems over a given logic L, as being the fusion of logics obtained from L and a set of normative systems over L, this model-theoretical construction will help us to understand better which properties are preserved under combinations of logics over which we have imposed some restrictions and to apply known transfer results (for a general account on the combination of logics, we refer to [4] and [5], and as a general reference on multimodal logic, to [6]). There are some advantages of using these logics for reasoning about MAS: it is possible to compare whether a normative system is more restrictive than the other, check if a certain property holds in a model of a logic once a normative system has restricted its accessibility relation, model the dynamics of normative systems in institutional settings, define a hierarchy of normative systems (and, by extension, a classification of the institutions) or present a logical-based reasoning model for the agents to negotiate over norms.

This paper is structured as follows. In Section 2 we present an example in order to motivate the introduction of the general framework. In Section 3 we give a sound and complete axiomatisation for the Multimodal Logic of Normative Systems, proving transfer results and we address a complexity issue for model checking. In Section 4 we restrict our attention to logics with normative systems that define elementary classes of modal frames, we have called them *Elementary Normative Systems (ENS)* and we prove completeness and canonicity results for them. Elementary classes include a wide range of formalisms used in describing MAS, modelling different aspects of agenthood, some temporal logics, logics of knowledge and belief, logics of communication, etc. Finally, in Section 5 we come back to our first example in Section 2, showing how our framework can be applied to multiprocess temporal structures, Section 6 is devoted to future work.

2 Multiprocess Temporal Frames and Normative Systems

In a multi-agent institutional environment, in order to allow agents to successfully interact with other agents, they share the dialogic framework. The expressions of the communication language in a dialogic framework are constructed as formulas of the type $\iota(\alpha_i : \rho_i, \alpha_j : \rho_j, \phi, \tau)$, where ι is an illocutionary particle, α_i and α_j are agent terms, ρ_i and ρ_j are role terms and τ is a time term. An scene is specified by a graph where the nodes of the graph represent the different states of the conversation and the arcs connecting the nodes are labelled with illocution schemes.

Several formalisms for modelling interscene exchanges between agents have been introduced using multimodal logics. For instance, in [7] the authors provide an alternating offers protocol to specify commitments that agents make to each other when engaging in persuasive negotiations using rewards. Specifically, the protocol details, how commitments arise or get retracted as a result of agents promising rewards or making offers. The protocol also standardises what an agent is allowed to say or what it can expect to receive from its opponent. The multimodal logic presented in [7] introduces modalities $\Box_\phi$ for expressions ϕ of the communication language.

More formally, given a finite set of propositional atomic formulas, we could define the set of formulas of such a multimodal communication language in the following way:

$$\phi ::= p \mid \top \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box_{\phi_1}\alpha \mid \ldots \mid \Box_{\phi_k}\alpha$$

where p is an atomic propositional formula, α is a propositional formula and $\phi_1, \ldots, \phi_k$ are formulas of the communication language.

The standard Kripke semantics of these logics can be given by means of multiprocess temporal frames. We say that $\Xi = (S, R_{\phi_0}, \ldots, R_{\phi_k})$ is a *multiprocess temporal frame* if and only if S is a set of states and for every $i \leq k$, R_{ϕ_i} is a binary relation on S such that $R = \bigcup_{i \leq k} R_{\phi_i}$ is a serial relation (that is, for every $s \in S$ there is $t \in S$ such that $(s, t) \in R$). A *multiprocess temporal model* is a Kripke model with a multiprocess temporal frame.

Let M be a multiprocess temporal model and $w \in M$, the satisfiability relation for the modalities $\Box\phi_i$ is defined as usual:

$$M, w \models \Box_{\phi_i}\alpha \text{ iff for all } w' \in M \text{ such that } wR_{\phi_i}w'$$

$$M, w' \models \alpha$$

Some examples of the protocols introduced in [7] can be formalised by formulas of the following form: $\Box_{\phi_1} \ldots \Box_{\phi_l}\bot$. For instance, with the formula $\Box_{Offer(i,x)}\Box_{Offer(i,y)}\bot$, with $x \neq y$, we can express that it is not allowed to agent i to do two different offers one immediately after the other. Let us see now how formulas like $\Box_{\phi_1} \ldots \Box_{\phi_l}\bot$ can be understood as sets of constraints on the transitions between states. Given a multiprocess temporal frame $\Xi = (S, R_{\phi_0}, \ldots, R_{\phi_k})$, consider the following set of finite sequences of elements of S:

$$\Delta_\Xi = \{(a_0, \ldots, a_m) : \forall j < m, \exists i \leq k \text{ such that } a_j R_{\phi_i} a_{j+1}\}$$

Then, a *normative system* η on the frame Ξ could be defined as a subset of Δ_Ξ. Intuitively speaking, a sequence $(a_0, \ldots, a_m) \in \eta$ if and only if this sequence of transitions is not legal according to normative system η. In our previous example, given a frame, the formula $\Box_{Offer(i,x)}\Box_{Offer(i,y)}\bot$, can be regarded as the following normative system (that is, the following set of finite sequences of the frame):

$$\{(a_0, a_1, a_2) : \text{ such that } a_0 R_{Offer(i,x)} a_1 \text{ and } a_1 R_{Offer(i,x)} a_2\}$$

Thus, any model satisfying the protocol introduced by $\Box_{Offer(i,x)}\Box_{Offer(i,y)}\bot$ can not include such sequences.

When defining an scene in an electronic institution we could be interested in comparing different protocols in order to show which of them satisfy some desired properties. In order to do so we could extend our multimodal language with additional modalities $\Box^{\eta}_{\phi_i}$, one for each normative system we want to consider. Next section is devoted to the study of the logical properties of these languages and later on, we will come back to our example applying this general framework.

3 Multimodal Logics of Normative Systems

We introduce first some notation and basic facts about multimodal languages. A *finite modal similarity type* $\tau = \langle F, \rho \rangle$ consists of a finite set F of modal operators and a map $\rho : F \to \omega$ assigning to each $f \in F$ a finite arity $\rho(f) \in \omega$. Finite propositional modal languages of type τ are defined in the usual way by using finitely many propositional variables, the operators in F and the boolean connectives $\wedge, \vee, \neg, \to, \leftrightarrow, \top, \bot$. For monadic modalities we use the usual notation $\Box_f$.

A *modal finitary structural consequence relation* $\vdash$ of similarity type τ is a relation between sets of formulas and formulas of the finite propositional modal language of type τ satisfying:

- $\phi \in \Gamma \Rightarrow \Gamma \vdash \phi$
- If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \phi$, then $\Delta \vdash \phi$
- If $\Gamma \vdash \Delta$ and $\Delta \vdash \phi$, then $\Gamma \vdash \phi$
- $\Gamma \vdash \phi \Rightarrow s\Gamma \vdash s\phi$, for all substitutions s
- If $\Gamma \vdash \phi$, then there exist a finite subset Γ_0 of Γ with $\Gamma_0 \vdash \phi$
- $\vdash \phi$, for every classical tautology ϕ
- $p, p \to q \vdash q$
- For every $f \in F$,

$$p_0 \leftrightarrow q_0, \ldots, p_{\rho(f)} \leftrightarrow q_{\rho(f)} \vdash f(p_0, \ldots, p_{\rho(f)}) \leftrightarrow f(q_0, \ldots, q_{\rho(f)})$$

And we say that a subset Λ of modal formulas is a *classical modal logic* of similarity type τ iff there exists a modal finitary structural consequence relation $\vdash$ of similarity type τ such that $\Lambda = \Lambda(\vdash)$, where $\Lambda(\vdash) = \{\phi : \emptyset \vdash \phi\}$. It is said that that Λ is *consistent* if $\bot \notin \Lambda$.

Given a type $\tau = \langle F, \rho \rangle$, a *Kripke frame* of type τ is an structure $(S, R_f)_{f \in F}$, where S is nonempty and for every $f \in F$, R_f is a binary relation on S.

Definition 1 *A normative system over a Kripke frame $(S, R_f)_{f \in F}$ is a subset of the following set of finite sequences of S:*

$$\{(a_0, \ldots, a_m) : \forall j < m, \exists f \in F \text{ such that } a_j R_f a_{j+1}\}$$

Observe that Definition 1 extends to the multimodal setting the definition of normative system introduced in Section 2 of [3]. Examples of classical modal logics with semantics based on Kripke frames are Propositional Dynamic Logic (PDL), Alternating-Time Temporal Logic (ATL) and Computational Tree Logic (CTL), but CTL*, the Full Computational Tree Logic is not a classical modal logic because it is not closed under uniform substitution.

Now we introduce in the language a new finite set of symbols N to denote normative systems. Given a finite propositional modal language of type $\tau = \langle F, \rho \rangle$, for every normative system $\eta \in N$, let τ^η be the type whose modalities are $\{f^\eta : f \in F\}$ and $\tau^N = \bigcup_{\eta \in N} \tau^\eta$. For every set of formulas Γ, let us denote by Γ^η the set of formulas of type τ^η obtained from Γ by substituting every occurrence of the modality f by f^η. The monadic operators $\Diamond_f$ are defined in the usual way as abbreviations $\Diamond_f \phi \equiv \neg \Box_f \neg \phi$ and we have also the corresponding $\Diamond_f^\eta$.

Given a classical modal logic L with semantics based on Kripke frames, we define the *Multimodal Logic of Normative Systems* over L, denoted by L^N, as being the smallest classical modal logic in the expanded language τ^N which contains L and L^η, for every $\eta \in N$.

Theorem 2 *Let L be a consistent classical modal logic axiomatised by a set Γ of formulas. Then,*

1. $\Gamma^N = \Gamma \cup \bigcup \{\Gamma^\eta : \eta \in N\}$ *is an axiomatisation of* L^N.
2. L^N *is a conservative extension of* L.
3. *If L is a decidable logic, then* L^N *is decidable.*

Proof: Since we have introduced a finite set of disjoint similarity types $\{\tau^\eta : \eta \in N\}$, we can define the fusion $\bigoplus < L^\eta : \eta \in N >$ of disjoint copies of the logic L. Observe that, so defined, $L^N = \bigoplus < L^\eta : \eta \in N >$ and Γ^N is an axiomatisation of L^N. Then, by an early result of Thomason [8], L^N is a conservative extension of L. Finally we can apply Theorem 6.11 of [9], to obtain the corresponding transfer result. $\Box$

In [10] a weak notion of normality is introduced to prove some additional transfer results for the fusion of logics. Let us assume that our classical modal logics satisfy the two conditions of Definition 2.5 of [10]:

1. For every $f \in F$, the semantics of $f(p_0, \ldots, p_{\rho(f)})$ is a monadic first-order formula.
2. For each R_f, there is a derived connective $\Box_f$ such that the formula $\Box_f p$ expresses $\forall x (y R_f x \rightarrow P x)$ and is closed under the necessitation rule: If $\phi \in \Lambda$, then $\Box_f \phi \in \Lambda$.

This second condition corresponds to the notion of normality, but it is weaker than the usual normality requirement. Observe that the operators U and S (until and since) of Temporal Logic are only normal in the first position and not in the second. However, they satisfy conditions 1. and 2., the binary ordering $<$ can be associated with U and the binary ordering $>$ can be associated with S, thus condition 1. is satisfied. The monadic modalities H and G are derivable connectives, that satisfy the requirement of condition 2.

Following the lines of the proof of Theorem 2, by using Theorems 3.6 and 3.10 of [10], we can obtain the following transfer theorem:

Theorem 3 *Let L be a consistent classical modal logic axiomatised by a set Γ of formulas and such that satisfies conditions 1. and 2. above. Then, If L is complete and sound over the class of frames C, then L^N is also complete and sound over the class of frames $\bigoplus < C^\eta : \eta \in N >$.*

As an application of Theorems 2 and 3 we obtain that the Multimodal Logic of Normative Systems over the logics CTL and PDL, has a sound and complete axiomatisation, is decidable and has the Finite Model Property, because CTL and PDL are decidable and complete over the class of finite frames.

We end this section by introducing a model checking result. Given a frame $\Xi = (S, R_f)_{f \in F}$, we say that a subset of S is *connected* if for every $s, t \in S$, $(s,t) \in (\bigcup \left\{ (R_f \cup R_f^{-1} : f \in F \right\})^*$, where for any relation R, R^* denotes the transitive closure of R. We say that the frame Ξ is connected if its domain S is a connected set. Observe that, for every classical modal logic L that satisfies conditions 1. and 2. stated above and it is complete with respect to a class of connected frames, by Theorem 3, the Multimodal Logic of Normative Systems over L is also complete with respect to a class of connected frames.

Theorem 4 *Let L be a classical modal logic in a finite similarity type $\tau = \langle F, \rho \rangle$ and let $(S, R_f^\eta)_{f \in F, \eta \in N}$ be a finite model of the Multimodal Logic of Normative Systems over L such that the restriction of the model $(S, R_f^\eta)_{f \in F, \eta \in N}$ to the similarity type τ^η is connected. Then, the complexity of model checking a formula ϕ of type τ^N is*

$$O(\textstyle\sum_{\eta \in N} m_\eta + n \cdot k) + \sum_{\eta \in N}((O(k) + O(n)) \cdot C_L(m_\eta, n, k))$$

where $m_\eta = \sum_{f \in F} \left| R_f^\eta \right|$, $n = |S|$, k is the length of the formula ϕ and $C_L(m_\eta, n, k)$ is the complexity of model checking for logic L as a function of m_η, n and k.

Proof: By Theorem 2, L^N is a conservative extension of L and for every $\eta \in N$ the restriction of the model $(S, R_f^\eta)_{f \in F, \eta \in N}$ to the similarity type τ^η is a model of L and is connected by assumption. This fact allows us to generalize the result on temporal logics of Theorem 5.2 of [11]. We can express the complexity of a combined model checker for L^N in terms of a model checker for L. $\Box$

For example, in the case of the Multimodal Logic of Normative Systems over CTL, the overall cost of the model checker for this logic is linear in the size of the model and in the length of the formula.

4 Elementary Normative Systems

There are some advantages of using Multimodal Logics of Normative Systems for reasoning about MAS: it is possible to compare whether a normative system is more restrictive than the other, check if a certain property holds in a model of a logic once a normative system has restricted its accessibility relation, model the dynamics of normative systems in institutional settings, define a hierarchy of normative systems (and, by extension, a classification of the institutions) or present a logical-based reasoning model for the agents to negotiate over norms. Up to this moment we have introduced an extensional definition of normative system (see Definition 1), in this section we present our first attempt to classify normative systems, we restrict our attention to normative systems defined by certain sets of first-order formulas, but only over some class of normal multimodal logics with standard Kripke semantics.

The choice of Sahlqvist formulas in this section is due, on the one hand, to the fact that a wide range of formalisms for MAS can be axiomatised by a set of such formulas (see next section). On the other hand, for the good logical properties of these logics (canonicity, transfer results, etc.). In Section 3 we have presented a general setting for dealing with any classical modal logic. Now, we focus only on some particular kind of logics. We want to study the specific properties of their normative systems that can be proved by using only the fact that these logics are axiomatised by sets of Sahlqvist formulas.

Given a set of modal formulas Σ, the *frame class defined by* Σ is the class of all frames on which each formula in Σ is valid. A frame class is *modally definable* if there is a set of modal formulas that defines it, and it is said that the frame class is *elementary* if it is defined by a first-order sentence of the frame correspondence language (the first-order language with equality and one binary relation symbol for each modality). An *Elementary Normative System* (ENS) is a propositional modal formula that defines an elementary class of frames and a normative system in any frame.

Throughout this and next section we assume that our modal languages have standard Kripke semantics and their modal similarity types have only a finite set of monadic modalities $\{\Box_f : f \in F\}$ and a finite set of propositional variables. Given a classical modal logic L and a set of Elementary Normative Systems N over L, for every $\eta \in N$ we generalize the notion introduced in Section 3 by defining the *Multimodal Logic of Normative Systems* over L and N, denoted by L^N, as being the smallest normal logic in the expanded language which contains L, N and every L^η. We now present a sound and complete axiomatisation and prove some transfer results in the case that L is axiomatised by a set of Sahlqvist formulas and N is a set of Sahlqvist formulas. We denote by $L(\eta)$ the smallest normal logic of similarity type τ^η which includes $L^\eta \cup \{\eta\}$.

Definition 5 (Sahlqvist formulas) *A modal formula is* positive (negative) *if every occurrence of a proposition letter is under the scope of an even (odd) number of negation signs. A* Sahlqvist antecedent *is a formula built up from* $\top$, $\bot$, *boxed atoms of the form* $\Box_{i_1} \dots \Box_{i_l} p$, *for* $i_j \in I$ *and negative formulas, using conjunction, disjunction and diamonds. A* Sahlqvist implication *is a formula of the form* $\phi \to \varphi$, *when* ϕ *is a Sahlqvist antecedent and* φ *is positive. A* Sahlqvist formula *is a formula that is obtained from Sahlqvist implications by applying boxes and conjunction, and by applying disjunctions between formulas that do not share any propositional letters.*

Observe that $\bot$ and $\top$ are both Sahlqvist and ENS formulas. Intuitively speaking, $\bot$ is the trivial normative system, in $\bot$ every transition is forbidden in every state and in $\top$ every action is legal. In the sequel we assume that for every set N of ENS, $\top \in N$.

Theorem 6 *Let L be a classical normal modal logic axiomatised by a set Γ of Sahlqvist formulas and N a set of ENS Sahlqvist formulas, then:*

1. *$\Gamma^N = \Gamma \cup N \cup \bigcup \{\Gamma^\eta : \eta \in N\}$ is an axiomatisation of L^N.*
2. *L^N is complete for the class of Kripke frames defined by Γ^N.*
3. *L^N is canonical.*
4. *If L and L^η are consistent, for every $\eta \in N$, and **P** is one of the following properties:*
   - *Compactness*
   - *Interpolation Property*
   - *Halldén-completeness*
   - *Decidability*
   - *Finite Model Property*[1]

   *then L^N has **P** iff L and $L(\eta)$ have **P**, for every $\eta \in N$.*

Proof: $1-3$ follows directly from the Sahlqvist's Theorem. The main basic idea of the proof of 4 is to apply the Sahlqvist's Theorem to show first that for every $\eta \in N$, the smallest normal logic of similarity type τ^η which includes $\Gamma^\eta \cup \{\eta\}$ is $L(\eta)$, is a complete logic for the class of Kripke frames defined by $\Gamma^\eta \cup \{\eta\}$ and is canonical (observe that this logic is axiomatised by a set of Sahlqvist formulas). Now, since for every Elementary Normative System $\eta \in N$ we have introduced a disjoint modal similarity type τ^η, we can define the fusion of the logics $\bigoplus < L(\eta) : \eta \in N >$. It is enough to check that $L^N = \bigoplus < L(\eta) : \eta \in N >$ (remark that $L^\top = L$) and using transfer results for fusions of consistent logics (see for instance [12] and [10]) we obtain that L^N is a conservative extension and that decidability, compactness, interpolation, Hállden-completeness and the Finite Model Property are preserved. $\Box$

We study now the relationships between normative systems. It is interesting to see how the structure of the set of all the ENS over a logic L (we denote it by $N(L)$) inherits its properties from the set of first-order counterparts. A natural relationship could be defined between ENS, the relationship of being one *less restrictive* than another, let us denote it by $\preceq$. Given η, η', it is said that $\eta \preceq \eta'$ iff the first-order formula $\phi_{\eta'} \to \phi_\eta$ is valid (when for every $\eta \in N$, ϕ_η is the translation of η). The relation $\preceq$ defines a partial order on $N(L)$ and the pair $(N(L), \preceq)$ forms a complete lattice with least upper bound $\perp$ and greatest lower bound $\top$ and the operations $\wedge$ and $\vee$.

Now we present an extension of the Logic of Elementary Normative Systems over a logic L with some inclusion axioms and we prove completeness and canonicity results. Given a set N of ENS, let I^{N^+} be the following set of formulas:

$$\left\{\Box_{i_1} \ldots \Box_{i_l} p \to \Box^\eta_{i_1} \ldots \Box^\eta_{i_l} p : i_j \in I, \eta \in N\right\}$$

and I^{N^*} the set:

$$\left\{\Box^{\eta'}_{i_1} \ldots \Box^{\eta'}_{i_l} p \to \Box^\eta_{i_1} \ldots \Box^\eta_{i_l} p : i_j \in I, \eta \preceq \eta', \eta, \eta' \in N\right\}$$

[1] For the transfer of the Finite Model Property it is required that there is a number n such that each $L(\eta)$ has a model of size at most n.

Corollary 7 *Let L be a normal modal logic axiomatised by a set Γ of Sahlqvist formulas and N a set of ENS Sahlqvist formulas, then:*

1. *$\Gamma^{N^+} = \Gamma^N \cup I^{N^+}$ is an axiomatisation of the smallest normal logic with contains L^N and the axioms I^{N^+}, is complete for the class of the Kripke frames defined by Γ^{N^+} and is canonical. We denote this logic by L^{N^+}.*
2. *$\Gamma^{N^*} = \Gamma^N \cup I^{N^*} \cup I^{N^+}$ is an axiomatisation of the smallest normal logic with contains L^N and the axioms $I^{N^*} \cup I^{N^+}$, is complete for the class of the Kripke frames defined by Γ^{N^*} and is canonical. We denote this logic by L^{N^*}.*
3. *If L^N is consistent, both L^{N^+} and L^{N^*} are consistent.*

Proof: Since for every $i_j \in I$ every $\eta, \eta' \in N$, the formulas $\Box_{i_1} \ldots \Box_{i_l} p \rightarrow \Box^{\eta}_{i_1} \ldots \Box^{\eta}_{i_l} p$ and $\Box^{\eta'}_{i_1} \ldots \Box^{\eta'}_{i_l} p \rightarrow \Box^{\eta}_{i_1} \ldots \Box^{\eta}_{i_l} p$ are Sahlqvist, we can apply Theorem 6. In the case that L^N is consistent, consistency is guaranteed by the restriction to pairs $\eta \preceq \eta'$ and for the fact that η and η' are ENS. $\Box$

Observe that for every frame $(S, R_f, R^{\eta}_f)_{f \in F, \eta \in N}$ of the logic L^{N^*},

$$R^{\eta}_{i_1} \circ \ldots \circ R^{\eta}_{i_l} \subseteq R_{i_0} \circ \ldots \circ R_{i_l},$$

and for $\eta \preceq \eta'$, $R^{\eta}_{i_1} \circ \ldots \circ R^{\eta}_{i_l} \subseteq R^{\eta'}_{i_1} \circ \ldots \circ R^{\eta'}_{i_1}$, where $\circ$ is the composition relation.

We end this section introducing a new class of modal formulas defining elementary classes of frames, the shallow formulas (for a recent account of the model theory of elementary classes and shallow formulas we refer the reader to [13]).

Definition 8 *A modal formula is* shallow *if every occurrence of a proposition letter is in the scope of at most one modal operator.*

It is easy to see that every closed formula is shallow and that the class of Sahlqvist and shallow formulas don't coincide: $\Box_1(p \vee q) \rightarrow \Diamond_2(p \wedge q)$ is an example of shallow formula that is not Sahlqvist. Analogous results to Theorem 6 and Corollary 7 hold for shallow formulas, and using the fact that every frame class defined by a finite set of shallow formulas admits polynomial filtration, by Theorem 2.6.8 of [13], if L is a normal modal logic axiomatised by a finite set Γ of shallow formulas and N is a finite set of ENS shallow formulas, then the frame class defined by Γ^N has the Finite Model Property and has a satisfiability problem that can be solved in NEXPTIME.

5 Some examples

Different formalisms have been introduced in the last twenty years in order to model particular aspects of agenthood (temporal Logics, logics of knowledge and belief, logics of communication, etc). We show in this section that several logics proposed for describing Multi-Agents Systems are axiomatised by a set of Sahlqvist or shallow formulas and therefore we could apply our results to the study of their normative systems. Let us come back to our previous example of Section 2, the multiprocess temporal frames. We have introduced first this basic temporal logic of transition systems, not because it is specially interesting in itself, but because is the logic upon which other temporal logics are built and because it is a clear and simple example of how our framework can work.

Remember that $\Xi = (S, R_0, \ldots, R_k)$ is a *multiprocess temporal frame* if and only if S is a set of states, for every $i \leq k$, R_i is a binary relation on S such that $R = \bigcup_{i \leq k} R_i$ is a serial relation (that is, for every $s \in S$ there is $t \in S$ such that $(s,t) \in R$). It is easy to see that $\Xi = (S, R_0, \ldots, R_k)$ is a multiprocess temporal frame if and only if the formula of the corresponding multimodal language

$$\Diamond_0 \top \vee \ldots \vee \Diamond_k \top \text{ (MPT)}$$

is valid in Ξ. Let us denote by $MPTL$ the smallest normal logic containing axiom (MPT). For every nonempty tuple $(i_1, \ldots, i_l)$ such that for every $j \leq l$, $i_j \leq k$, consider the formula $\Box_{i_1} \ldots \Box_{i_l} \bot$. Observe that every formula of this form is shallow and ENS. We state now without proof a result on the consistency of this kind of normative systems over $MPTL$ that will allow us to use the logical framework introduced in the previous section.

Proposition 9 *Let N be a finite set of normative systems such that for every $\eta \in N$, there is a finite set X of formulas of the form $\Box_{i_1} \ldots \Box_{i_l} \bot$ such that η is the conjunction of all the formulas in X, $\bot \notin X$ and the following property holds:*

$$\textit{If } \Box_{i_1} \ldots \Box_{i_l} \bot \notin X, \textit{ there is } j \leq k \textit{ such that } \Box_{i_1} \ldots \Box_{i_l} \Box_j \bot \notin X.$$

Then, the logic $MPTL^N$ is consistent, complete, canonical, has the Finite Model Property and has a satisfiability problem that can be solved in NEXPTIME.

In general, a normal multimodal logic can be characterized by axioms that are added to the system K_m, the class of *Basic Serial Multimodal Logics* is characterized by subsets of axioms of the following form, requiring that AD(i) holds for every i,

- $\Box_i p \to \Diamond_i p$ AD(i)
- $\Box_i p \to p$ AT(i)
- $\Box_i p \to \Box_j p$ AI(i)
- $p \to \Box_i \Diamond_j p$ AB(i,j)
- $\Box_i p \to \Box_j \Box_k p$ A4(i,j,k)
- $\Diamond_i p \to \Box_j \Diamond_k p$ A5(i,j,k)

An example of a Kripke frame of $MPTL$ in which none of the previous axioms is valid is $\Xi = (\{0,1,2\}, \{(0,1),(2,0)\}, \{(1,2)\})$. In particular, our example shows that the Multimodal Serial Logic axiomatised by $\{AD(i) : i \leq k\}$, is a proper extension of $MPTL$. Observe that any logic in the class BSML is axiomatised by a set of Sahlqvist formulas, therefore we could apply the framework introduced before to compare elementary normative systems on these logics.

Another type of logics axiomatised by Sahlqvist formulas are many Multimodal Epistemic Logics. Properties such as positive or negative introspection can be expressed by $\Box_i p \to \Box_i \Box_k p$ and $\neg\Box_i p \to \Box_i \neg\Box_i p$ respectively. And formulas like $\Box_i p \to \Box_j p$ allow us to reason about multi-degree belief.

The Minimal Temporal Logic K_t is axiomatised by the axioms $p \to HFp$ and $p \to GPp$ which are also Sahlqvist formulas. Some important axioms such as linearity $Ap \to GHp \wedge HGp$, or density $GGp \to Gp$, are Sahlqvist formulas, and we can express the property that the time has a beginning with an ENS. By adding the next-time modality, X, we have an ENS which expresses that every instant has at most one immediate successor.

6 Future work

Along this work, in Sections 4 and 5, we have dealt only with multimodal languages with monadic modalities, but by using the results of Goranko and Vakarelov in [14] on the extension of the class of Sahlqvist formulas in arbitrary polyadic modal languages to the class of inductive formulas, it would be possible to generalize our results to polyadic languages.

We will proceed to apply our results to different extended modal languages, such as reversive languages with nominals (in [14], the elementary canonical formulas in these languages are characterized) or Hybrid Logic (in [13], Hybrid Sahlqvist formulas are proved to define elementary classes of frames). Future work should go beyond Elementary Normative Systems and consider the study of sets of normative systems expressed by other formal systems.

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